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ON THE PATH DERIVATIVE

1. Introduction

Bruckner, O'Malley and Thomson in [3] introduced the concept of the path derivative as a unifying approach to the study of a number of generalized derivatives. Many proofs are based on a system of paths satisfying some of the intersection conditions. On the other hand the paper [1] uses the system of paths as a continuous multifunction. Our paper is a continuation of this approach. We study various properties of primitives and the properties of the multifunction of the E-derived numbers and the extreme path derivatives (namely the Baire classification). Our proofs are based on various generalized types of continuity and measurability of the system of paths.

2. The $\mathcal{E}$-derivative

Given nonempty sets X and Y, a function $F : X \longrightarrow 2^Y$, called a relation, and $A \subset Y$ let

$$F^-(A) = \{x \in X: F(x) \cap A \neq \phi\}$$

$$F^+(A) = \{x \in X: F(x) \subset A\}$$

$$GrF = \{(x,y) \in X\text{x}Y: x \in X, y \in F(x)\} \qquad \text{(graph of } F)$$

If for each $x \in X$ $F(x) \neq \phi$, then F is called a multifunction and we write $F : X \longrightarrow Y$.

Let $(\mathbb{R},\mathcal{O})$, $(\mathbb{R}^*,\mathcal{O}^*)$ be the real line with the usual topology and the extended real line with the topology of the two-point compactification of $\mathbb{R}$ respectively.

Definition 2.1. Let $(\mathbb{R},\mathcal{T})$ be a topological space, $\mathcal{O} \subset \mathcal{T}$. A quadruple $\mathcal{E} = (\mathbb{R},\mathcal{T},E,\mathcal{C})$ is called a generalized system of paths, where $E : \mathbb{R} \longrightarrow \mathbb{R}$ is multifunction, $\phi \neq \mathcal{C} \subset 2^{\mathbb{R}}$, $\phi \notin \mathcal{C}$.

Definition 2.2. Let $\mathcal{E}$ be a generalized system of paths and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. A point $z \in \mathbb{R}^*$ is called a $\mathcal{E}$-derived number of f at point $x \in \mathbb{R}$, if for any $G \in \mathcal{O}^*$, $z \in G$ and for any $U \in \mathcal{T}$, $x \in U$ there exists a set $A \in \mathcal{C}$ such that $A \subset U \cap E(x)\setminus\{x\}$ and for any $y \in A$: $\frac{f(x) - f(y)}{x - y} \in G$. The set of all $\mathcal{E}$-derived numbers of f at a point x will be denoted by $D(f,\mathcal{E},x)$.

Define $D_{f,\mathcal{E}} : \mathbb{R} \longrightarrow \mathbb{R}^*$ by $D_{f,\mathcal{E}}(x) = D(f,\mathcal{E},x)$. If $D(f,\mathcal{E},x) \neq \phi$, then the extreme $\mathcal{E}$-derivatives of f at a point x are

$\overline{f}'_{\mathcal{E}}(x) = \sup D(f,\mathcal{E},x)$ (the upper extreme $\mathcal{E}$-derivative)

$\underline{f}'_{\mathcal{E}}(x) = \inf D(f,\mathcal{E},x)$ (the lower extreme $\mathcal{E}$-derivative)

If $D(f,\mathcal{E},x)$ is a one point set, then that point is called the $\mathcal{E}$-derivative of f at x and it is denoted by $f'_{\mathcal{E}}(x)$.

Let $\ker(f,\mathcal{E}) = D_{\overline{f},\mathcal{E}}(\mathbb{R}) = \{x \in \mathbb{R} : f$ has at least one finite $\mathcal{E}$-derivative number$\}$.

Remark 2.3. If there is a set $U \in \mathcal{T}$ such that $x \in U$ and $U\setminus\{x\}$ does not contain any set from $\mathcal{C}$, then $D(f,\mathcal{E},x) = \phi$.

Definition 2.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ be a function and $\phi \neq \mathcal{C} \subset 2^{\mathbb{R}}$, $\phi \notin \mathcal{C}$. The $(\mathcal{C},\mathcal{T})$-cluster set, $C(\mathcal{T},f,x)$ of f at $x \in \mathbb{R}$ is the set of all points $y \in \mathbb{R}^*$ such that for all $V \in \mathcal{T}$ with $x \in V$ and all $U \in \mathcal{O}^*$ with $y \in U$, $f^{-1}(U) \cap V$ contains a set $A \in \mathcal{C}$. A function f is said to be $(\mathcal{C},\mathcal{T})$-continuous at a point $x \in \mathbb{R}$ if $f(x) \in C(\mathcal{T},f,x)$ and $(\mathcal{C},\mathcal{T})$-continuous if f is $(\mathcal{C},\mathcal{T})$-continuous at any $x \in \mathbb{R}$. A function $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ is said to be $\mathcal{T}$-measurable if $f^{-1}(G)$ has the $\mathcal{T}$-Baire property for any $G \in \mathcal{O}^*$ ([5, p. 306]).

Lemma 2.5. Let $\mathcal{E} = (\mathbb{R},\mathcal{T},E,\mathcal{C})$ be a generalized system of paths and let $f : R \rightarrow R$. If $x \in \ker(f,\mathcal{E})$, then f is $(\mathcal{C},\mathcal{T})$-continuous at x.

<u>Proof</u>. If $x \in \ker(f,\mathcal{E})$, then there exists $a \in \mathbb{R} \cap D(f,\mathcal{E},x)$. Let $U \in \mathcal{T}$, $x \in U$ and $\varepsilon > 0$. Let $M = \max\{|a - \varepsilon|, |a + \varepsilon|\}$ and $\delta < \varepsilon/M$. By Definition 2.2 there exists a set $A \subset (x - \delta, x + \delta) \cap U \cap E(x)\setminus\{x\}$ such

that $a - \varepsilon < \frac{f(x) - f(y)}{x - y} < a + \varepsilon$ for any $y \in A$. Hence for any $y \in A$ we have $\left|\frac{f(x) - f(y)}{x - y}\right| < M$. Thus $|f(x) - f(y)| < M|x - y| < M\delta < \varepsilon$ and thus f is $(\mathcal{C},\mathcal{T})$-continuous at x.

We turn now to a study of properties of the $\mathcal{E}$-primitives.

Theorem 2.6. Let $\mathcal{E} = (\mathbb{R},\mathcal{T},E,\mathcal{C}_0)$ be a generalized system of paths, where $\mathcal{C}_0 = \mathcal{T}\setminus\{\phi\}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$. If $\ker(f,\mathcal{E}) = \mathbb{R}$, then f is $\mathcal{T}$-quasicontinuous.

Proof. If $\mathcal{C}_0 = \mathcal{T}\setminus\{\phi\}$, then $(\mathcal{C}_0,\mathcal{T})$-continuity means $\mathcal{T}$-quasicontinuity. (See the originial definition in [4].). The assertion follows directly from Lemma 2.5.

Theorem 2.7. Let $\mathcal{E} = (\mathbb{R},\mathcal{T},E,\mathcal{C}_1)$ be a generalized system of paths, where $\mathcal{C}_1 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{T}$-second category and A has the $\mathcal{T}$-Baire property$\}$ (See [5, p. 54].) and let $f : \mathbb{R} \longrightarrow \mathbb{R}$. The following assertions are true.

(a) If $\ker(f,\mathcal{E}) = \mathbb{R}$, then f is $\mathcal{T}$-quasicontinuous.

(b) If $\ker(f,\mathcal{E})$ is $\mathcal{T}$-dense, then f is $\mathcal{T}$-measurable.

Proof. (a) By Lemma 2.5, f is $(\mathcal{C}_1,\mathcal{T})$-continuous. By Theorem 2.5 of [7], f is $\mathcal{T}$-quasicontinuous.

(b) Since $\ker(f,\mathcal{E})$ is $\mathcal{T}$-dense and since $\mathcal{C}_1(\mathcal{T},f,x) \neq \phi$ for any $x \in \ker(f,\mathcal{E})$, Theorem 5.5 of [7] implies that f is $\mathcal{T}$-measurable.

Theorem 2.8. Let $\mathcal{E} = (\mathbb{R},\mathcal{T},E,\mathcal{C}_2)$ be a generalized system of paths, where $\mathcal{C}_2 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{T}$-second category$\}$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is $\mathcal{T}$-measurable and $\ker(f,\mathcal{E}) = \mathbb{R}$, then f is $\mathcal{T}$-quasicontinuous.

Proof. By Lemma 2.5 f is $(\mathcal{C}_2,\mathcal{T})$-continuous. Since f is $\mathcal{T}$-measurable, f is $(\mathcal{C}_1,\mathcal{T})$-continuous, where $\mathcal{C}_1$ is as in Theorem 2.7. By Theorem 2.5 of [7], f is $\mathcal{T}$-quasicontinuous.

Remark 2.9. If $\mathcal{T} = \mathcal{N}$ is the density topology, then Theorem 2.7b gives Lebesgue measurability. (See [11].)

Remark 2.10. If $\mathcal{E} = (\mathbb{R},\mathcal{O},E,2^{\mathbb{R}}\setminus\{\phi\})$ and if each set $E(x)$ has x as a point of $\mathcal{O}$-accumulation, then we obtain the notion of the path derivative as defined in [3]. In this case we write $D(f,E)$, $\overline{f}'_E(x)$, $\underline{f}'_E(x)$, $f'_E(x)$ instead of $D(f,\mathcal{E})$, $\overline{f}'_{\mathcal{E}}(x)$, $\underline{f}'_{\mathcal{E}}(x)$, $f'_{\mathcal{E}}(x)$ and they are called, the set of all E-derived numbers of f at x, the upper, lower extreme E-derivative and the E-derivative of f at x, respectively.

Theorem 2.11. Suppose that $E(x)$ is of the $\mathcal{O}$-second category at x and $E(x)$ has the $\mathcal{O}$-Baire property for each $x \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$.

(a) If f has a finite E-derivative everywhere in an $\mathcal{O}$-dense set A, then f is $\mathcal{O}$-measurable.

(b) If f has a finite E-derivative everywhere, then f is $\mathcal{O}$-quasicontinuous.

Proof. It is clear that if f has a finite E-derivative at x, then f is E-continuous at x. Hence f is $(\mathcal{C}_1,\mathcal{O})$-continuous at x, where $\mathcal{C}_1 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{O}$-second category and A has the $\mathcal{O}$-Baire property$\}$.

In the case (a) $\mathcal{C}_1(\mathcal{O},f,x)$ is nonempty for each $x \in A$ and by Theorem 5.5 of [7] f is $\mathcal{O}$-measurable.

In the case (b) f is $(\mathcal{C}_1,\mathcal{O})$-continuous and by Theorem 2.5 of [7], f is $\mathcal{O}$-quasicontinuous.

3. The $\mathcal{E}s$-derivative

Mišík in [8] posed the following question. If the extreme, unilateral, essential derivative of a function f is almost everywhere finite, then is f Lebesgue measurable? In this section we shall show that the answer is yes and Theorem 3.10 will generalize this result. Throughout this section $(\mathbb{R},\mathcal{T})$ is Hausdorff topological space having no $\mathcal{T}$-isolated points. Let $D_{\mathcal{T}}(A) = \{x \in \mathbb{R}: V \cap A$ is of the $\mathcal{T}$-second category for any $V \in \mathcal{T}$, $x \in V\}$ and let $\text{int}_{\mathcal{T}}A$ denote the interior of A relative to $\mathcal{T}$. The following definition gives the topological analogue of essential derivatives.

Definition 3.1. Let $E : \mathbb{R} \longrightarrow \mathbb{R}$ be a multifunction. The triplet $(\mathbb{R},\mathcal{T},E)$ will be denoted by $\mathcal{E}s$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$, $x \in \mathbb{R}$. A point $z \in \mathbb{R}^*$ is called a $\mathcal{E}s$-derived number of f at x if $x \in D_{\mathcal{T}}(E(x))$ and for any $G \in \mathcal{O}^*$, $z \in G$

there is a $H \in \mathcal{T}$ containing x and there is a set $A \subset E(x) \cap H \setminus \{x\}$ such that $E(x) \cap H \setminus D_{\mathcal{T}}(A)$ is of the $\mathcal{T}$-first category and for any $y \in A$ $\frac{f(x) - f(y)}{x - y} \in G$. The set of all $\mathcal{E}s$-derived numbers of f at x will be denoted by $D(f,\mathcal{E}s,x)$ and $\overline{f}'_{\mathcal{E}s}$, $\underline{f}'_{\mathcal{E}s}$, $f'_{\mathcal{E}s}$ are defined analogously to Definition 2.2.

Example 3.2. Let $\mathcal{E}s = (\mathbb{R},\mathcal{N},E)$, where $\mathcal{N}$ is the density topology and $E(x) = \mathbb{R}$ for any $x \in \mathbb{R}$. If $a \in D(f,\mathcal{E}s,x)$, then for any $G \in \mathcal{O}^*$, $a \in G^*$ there is a set $A \subset \mathbb{R}$ such that $\limsup_{h\to 0} |A \cap (x - h, x + h)|^*/2h = 1$ and $\frac{f(x) - f(y)}{x - y} \in G$ for any $y \in A\setminus\{x\}$, where $|S|^*$ is the outer Lebesque measure of S.

Proof. It is clear that if $A \subset G \in \mathcal{N}$ and $D_{\mathcal{N}}(A) \supset G$, then $|A|^* = |G|$, where $|G|$ is the Lebesgue measure of G. If $a \in D(f,\mathcal{E}s,x)$, then for any $G \in \mathcal{O}^*$, $a \in G$ there is $H \in \mathcal{N}$, $x \in H$ and there is $A \subset E(x) \cap H\setminus\{x\} = H\setminus\{x\}$ such that $H\setminus D_{\mathcal{N}}(A)$ is of the $\mathcal{N}$-first category, i.e. $|H\setminus D_{\mathcal{N}}(A)| = 0$ and $\frac{f(x) - f(y)}{x - y} \in G$ for any $y \in A$. We shall show that $\limsup_{h\to 0} |A \cap (x - h, x + h)|^*/2h = 1$. From the following inclusions

$$((A \cap (x - h, x + h) \cap H)\setminus(H\setminus D_{\mathcal{N}}(A))) \subset (((x - h, x + h) \cap H)\setminus(H\setminus D_{\mathcal{N}}(A))) \in \mathcal{N}.$$

$$D_{\mathcal{N}}((A \cap (x - h, x + h) \cap H)\setminus(H\setminus D_{\mathcal{N}}(A))) \supset ((x - h, x + h) \cap H\setminus(H\setminus D_{\mathcal{N}}(A)))$$

we have

$$\limsup_{h\to 0} |A \cap (x - h, x + h)|^*/2h \geq$$

$$\limsup_{h\to 0} |(A \cap (x - h, x + h) \cap H)\setminus(H\setminus D_{\mathcal{N}}(A))|^*/2h =$$

$$\limsup_{h\to 0} |(x - h, x + h) \cap H\setminus(H\setminus D_{\mathcal{N}}(A))|/2h =$$

$$\limsup_{h\to 0} |(x - h, x + h) \cap H|/2h = 1 \quad \text{because} \quad x \in H \in \mathcal{N}.$$

Remark 3.3. In Example 3.2 if we put $E(x) = [x,\infty)$ ($E(x) = (-\infty,x]$) for any $x \in \mathbb{R}$, then we have the following:

If $a \in D(f,\mathcal{E}s,x)$, then

$$\limsup_{h\to 0} |A \cap (x,x+h)|^*/h = 1$$

$$\left(\limsup_{h\to 0} |A \cap (x-h,x)|^*/h = 1\right)$$

for any $G \in \mathcal{O}^*$, $a \in G$, where

$$A = \{y \in \mathbb{R}: \frac{f(x)-f(y)}{x-y} \in G, \quad y \neq x\}.$$

Definition 3.4. A multifunction $F : \mathbb{R} \longrightarrow \mathbb{R}^*$ is said to be upper semicontinuous (usc) (lower semicontinuous (lsc)) at a point $p \in \mathbb{R}$ if for any $G \in \mathcal{O}^*$ such that $G \supset F(p)$ ($G \cap F(p) \neq \phi$) there exists a set $V \in \mathcal{T}$ such that $p \in V$ and $F(x) \subset G$ ($F(x) \cap G \neq \phi$) for any $x \in V$. Let $\phi \neq \mathcal{C} \subset 2^{\mathbb{R}}$, $\phi \notin \mathcal{C}$. A multifunction $F : \mathbb{R} \longrightarrow \mathbb{R}^*$ is said to be ℓ-$(\mathcal{C},\mathcal{T})$-continuous at p if for any $G \in \mathcal{O}^*$ such that $G \cap F(p) \neq \phi$ and for any $V \in \mathcal{T}$ with $p \in V$ there is a set $A \in \mathcal{C}$ such that $A \subset V$ and $F(x) \cap G \neq \phi$ for any $x \in A$. If F is usc, (lsc, ℓ-$(\mathcal{C},\mathcal{T})$-continuous) at each $p \in \mathbb{R}$, then F is said to be usc (lsc, ℓ-$(\mathcal{C},\mathcal{T})$-continuous).

Remark 3.5. (a) In the case of a single valued multifunction ℓ-$(\mathcal{C},\mathcal{T})$-continuity coincides with the notion of $(\mathcal{C},\mathcal{T})$-continuity. (See Definition 2.4.)

(b) If $\mathcal{C} = \mathcal{T}\setminus\{\phi\}$, then ℓ-$(\mathcal{C},\mathcal{T})$-continuity of F means $\mathcal{T}$-lower quasisemicontinuity of F (lqsc).

Lemma 3.6. If $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ is an arbitrary function and $\mathcal{C}_2 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{T}$-second category$\}$, then the set K of all points at which f is not $(\mathcal{C}_2,\mathcal{T})$-continuous is of the $\mathcal{T}$-first category.

<u>Proof</u>. Denote by $\{G_n\}_{n=1}^{\infty}$ a countable base of $(\mathbb{R}^*,\mathcal{O}^*)$. We have $K = \bigcup_{n=1}^{\infty} (f^{-1}(G_n)\setminus D_{\mathcal{T}}(f^{-1}(G_n)))$ and according to [5, p. 51] the set K is of the $\mathcal{T}$-first category.

Lemma 3.7. Let $(\mathbb{R},\mathcal{T})$ be a Baire space. A function $f : \mathbb{R} \longrightarrow \mathbb{R}^*$ is $\mathcal{T}$-

nonmeasurable if and only if there are $a,b \in \mathbb{R}$, $a < b$ and $\phi \neq H \in \mathcal{T}$ such that $D_{\mathcal{T}}(f^{-1}((b,\infty])) \supset H$ and $D_{\mathcal{T}}(f^{-1}([-\infty,a))) \supset H$.

Proof. Suppose that f is not $\mathcal{T}$-measurable and let $\mathcal{C}_2 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{T}$-second category$\}$. By Lemma 3.6 the set K of all points at which f is not $(\mathcal{C}_2,\mathcal{T})$-continuous is of the $\mathcal{T}$-first category. Hence $\mathcal{C}_2(f,\mathcal{T},x)$ is nonempty on the $\mathcal{T}$-dense set $\mathbb{R}\setminus K$ because $f(x) \in \mathcal{C}_2(f,\mathcal{T},x)$ for any $x \in \mathbb{R}\setminus K$. Since $(\mathbb{R}^*,\mathcal{O}^*)$ is compact, the set $\mathcal{C}_2(f,\mathcal{T},x)$ is nonempty for any $x \in \mathbb{R}$, as is now shown. Let $x \in \mathbb{R}$. There is a net $\{x_\sigma \in \mathbb{R}\setminus K : \sigma \in \Sigma\}$ which converges to x. Since $(\mathbb{R}^*,\mathcal{O}^*)$ is compact, there is a subnet $\{y_{\sigma^1} \in \mathcal{C}_2(f,\mathcal{T},x_{\sigma^1}) : \sigma^1 \in \Sigma^1 \subset \Sigma\}$ which converges to $y \in \mathbb{R}^*$. It is clear that $y \in \mathcal{C}_2(f,\mathcal{T},x)$. Hence $\mathcal{C}_2(f,\mathcal{T},x) \neq \phi$.

Define $C : \mathbb{R} \longrightarrow \mathbb{R}^*$ as follows:
$C(x) = \mathcal{C}_2(f,\mathcal{T},x)$ for any $x \in \mathbb{R}$. By [7, Lemma 4.1] $C(x)$ is $\mathcal{O}^*$-compact for any $x \in \mathbb{R}$ and by [7, Lemma 4.4] C is usc. Hence C is lsc except for a set K_1 of the $\mathcal{T}$-first category (Theorem 2.1 of [7]).

We shall show that C is ℓ-$(\mathcal{C}_2,\mathcal{T})$-continuous. Let $x \in \mathbb{R}$, $G \in \mathcal{O}^*$ such that $G \cap C(x) \neq \phi$, let $U \in \mathcal{T}$, $x \in U$. By Definition 2.4, there is a set $A \in \mathcal{C}_2$ such that $A \subset U$ and $f(A) \subset G$. Since $f(z) \in \mathcal{C}_2(f,\mathcal{T},z)$ for any $z \in A\setminus K$, $\mathcal{C}_2(f,\mathcal{T},z) \cap G \neq \phi$ for any $z \in A\setminus K \in \mathcal{C}_2$. Thus C is ℓ-$(\mathcal{C}_2,\mathcal{T})$-continuous. By [7, Theorem 1.1] C is lqsc. By [7, Corolary 2 of Theorem 5.6] $M = \{x \in \mathbb{R} : C(x) \neq \{f(x)\}\}$ is of the $\mathcal{T}$-second category. Hence $C(x) \supsetneqq \{f(x)\}$ for any $x \in M \cap (\mathbb{R}\setminus K)$. Define $f_1,f_2 : \mathbb{R} \longrightarrow \mathbb{R}^*$ as follows: $f_1(x) = \sup C(x)$ and $f_2(x) = \inf C(x)$. Since C is usc, $f_2^{-1}((a,\infty]) \in \mathcal{T}$ and $f_1^{-1}((-\infty,a)) \in \mathcal{T}$ for any $a \in \mathbb{R}$. We shall show that f_1 and f_2 are $\mathcal{T}$-continuous except for a set of the $\mathcal{T}$-first category. We introduce the following notion of continuity: A function $g : (\mathbb{R},\mathcal{T}) \longrightarrow (\mathbb{R}^*,\mathcal{O}^*)$ is said to be H-continuous at x if $x \in \mathrm{int}_{\mathcal{T}} D_{\mathcal{T}}(g^{-1}(G))$ for any $G \in \mathcal{O}^*$, $g(x) \in G$. By Remark 1.1 of [7], g is H-continuous except for a set of the $\mathcal{T}$-first category. It is clear that if $g^{-1}((a,\infty]) \in \mathcal{T}$ (or $g^{-1}([-\infty,a)) \in \mathcal{T}$) for any $a \in \mathbb{R}$ (that is, g is $\mathcal{T}$-lower (upper) semicontinuous) and g is H-continuous at a point x, then g is $\mathcal{T}$-upper (lower) semicontinuous at x (that is, $x \in \mathrm{int}_{\mathcal{T}}(g^{-1}(G))$ for any $G \in \mathcal{O}^*$, $g(x) \in G$). Hence there is a set P such that $\mathbb{R}\setminus P$ is of the $\mathcal{T}$-first category and f_1 and f_2 are $\mathcal{T}$-continuous at x for any $x \in P$. Let $x_0 \in (\mathrm{int}_{\mathcal{T}} D_{\mathcal{T}}(M \cap (\mathbb{R}\setminus K))) \cap M \cap (\mathbb{R}\setminus K) \cap P$. Since $f_1(x_0) > f_2(x_0)$, there are $a,b \in \mathbb{R}$ such that $f_1(x_0) > b > a > f_2(x_0)$. The functions

f_1, f_2 are $\mathcal{T}$-continuous at x_0. Hence there is a nonempty set $H \in \mathcal{T}$ such that $H \subset \mathrm{int}_{\mathcal{T}} D_{\mathcal{T}}(M \cap (\mathbb{R}\setminus K))$ and $f_1(p) > b > a > f_2(p)$ for any $p \in H$. We shall show that $H \subset D_{\mathcal{T}}(f^{-1}((b,\infty])) \cap D_{\mathcal{T}}(f^{-1}([-\infty,a)))$. Let $z \in H$, $V \in \mathcal{T}$, $z \in V$. Let $t \in H \cap V\setminus K$. Then $f_1(t) > b$, $f_2(t) < a$, $f_1(t), f_2(t) \in C(t) = \mathcal{C}_2(f,\mathcal{T},t)$. Hence there are $A_1, A_2 \subset H \cap V$; $A_1, A_2 \in \mathcal{C}_2$, $A_1 \subset f^{-1}((b,\infty])$, $A_2 \subset f^{-1}([-\infty,a))$. Consequently $z \in D_{\mathcal{T}}(f^{-1}((b,\infty])) \cap D_{\mathcal{T}}(f^{-1}([-\infty,a)))$.

Suppose that f is $\mathcal{T}$-measurable. Then for any $a,b \in \mathbb{R}$, $a < b$ the sets $B = f^{-1}((b,\infty])$ and $A = f^{-1}([-\infty,a))$ have the $\mathcal{T}$-Baire property. By [5, p. 56], $D_{\mathcal{T}}(B)\setminus B$, $D_{\mathcal{T}}(A)\setminus A$ are of the $\mathcal{T}$-first category. Since $A \cap B = \phi$, $D_{\mathcal{T}}(B) \cap D_{\mathcal{T}}(A)$ is of the $\mathcal{T}$-first category. Therefore, there is no nonempty set $H \in \mathcal{T}$ such that $H \subset D_{\mathcal{T}}(B) \cap D_{\mathcal{T}}(A)$.

Definition 3.8. A multifunction E is said to be right sided (left sided) at x if $E(x) \subset [x,\infty)$ ($E(x) \subset (-\infty,x]$). E is right sided (left sided) on $T \subset \mathbb{R}$ if E is right sided (left sided) at each $x \in T$. E is unilateral on T if E is right sided on T or left sided on T.

Lemma 3.9. Let $(\mathbb{R},\mathcal{T})$ be a Baire space and $\mathcal{O} \subset \mathcal{T}$. Let E be unilateral on a $\mathcal{T}$-residual set T, $x \in D_{\mathcal{T}}(E(x))$ and let $E(x)$ have the $\mathcal{T}$-Baire property for each $x \in T$. If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not $\mathcal{T}$-measurable, then there are sets $S_1, S_2 \subset \mathbb{R}$ of the $\mathcal{T}$-second category such that $\overline{f}'_{\mathcal{E}S}(x) = \infty$ for any $x \in S_1$ and $\underline{f}'_{\mathcal{E}S}(x) = -\infty$ for any $x \in S_2$.

<u>Proof</u>. Suppose that E is right sided on T. (If E is left sided, the proof is analogous.) By Lemma 3.7 there are $a,b \in \mathbb{R}$, $a < b$ and there is a nonempty set $H \in \mathcal{T}$ such that $D_{\mathcal{T}}(B) \supset H$ and $D_{\mathcal{T}}(A) \supset H$, where $B = f^{-1}((b,\infty))$ and $A = f^{-1}((-\infty,a))$. Let $S_1 = A \cap H \cap T$. We shall show that $\overline{f}'_{\mathcal{E}S}(x) = \infty$ for any $x \in S_1$. Let $x \in S_1$, $c \in \mathbb{R}$. It is clear that there is $\delta > 0$ such that $(b - a)/(y - x) > c$ for any $y \in (x,x + \delta)$. Let $A_0 = B \cap E(x) \cap (x,x + \delta) \cap H$ and $H_0 = (x - \delta,x + \delta) \cap H \in \mathcal{T}$. We shall show that $D_{\mathcal{T}}(A_0) \supset D_{\mathcal{T}}(H_0 \cap E(x))$. Let $p \in D_{\mathcal{T}}(H_0 \cap E(x))$ and $V \in \mathcal{T}$, $p \in V$. Then $V \cap H_0 \cap E(x) = V \cap (x - \delta,x + \delta) \cap H \cap E(x) = V \cap [x,x + \delta) \cap H \cap E(x)$ is of the $\mathcal{T}$-second category. Hence $V \cap (x,x + \delta) \cap H \cap E(x)$ is of the $\mathcal{T}$-second category and $V \cap (x,x + \delta) \cap H \cap \mathrm{int}_{\mathcal{T}} D_{\mathcal{T}}(E(x))$ is nonempty. Since $E(x)$ has the $\mathcal{T}$-Baire property and $D_{\mathcal{T}}(B) \supset H \supset V \cap (x,x + \delta) \cap H \cap \mathrm{int}_{\mathcal{T}} D_{\mathcal{T}}(E(x))$, the set $B \cap V \cap (x,x + \delta) \cap H \cap E(x)$ is of the $\mathcal{T}$-second category and therefore $p \in D_{\mathcal{T}}(A_0)$.

Since $E(x) \cap H_0 \backslash D_{\mathcal{J}}(E(x) \cap H_0)$ is of the $\mathcal{J}$-first category (See [5, p. 51].), $E(x) \cap H_0 \backslash D_{\mathcal{J}}(A_0)$ is also of the $\mathcal{J}$-first category. Thus for any $c \in \mathbb{R}$ there is $H_0 \in \mathcal{J}$, $x \in H_0$ and there is $A_0 \subset E(x) \cap H_0 \backslash \{x\}$ such that $E(x) \cap H_0 \backslash D_{\mathcal{J}}(A_0)$ is of the $\mathcal{J}$-first category and for any $y \in A_0$, $c < (b-a)/(y-x) < (f(y) - f(x))/(y-x)$ because $f(y) > b$, $f(x) < a$. Hence $\overline{f}'_{\mathcal{E}S}(x) = \infty$.

Let $S_2 = B \cap H \cap T$. Let $x \in S_2$, $c \in \mathbb{R}$. It is clear that there is $\delta > 0$ such that $(b-a)/(x-y) < c$ for any $y \in (x, x + \delta)$. Let $A'_0 = A \cap E(x) \cap (x, x + \delta) \cap H$, $H_0 = (x - \delta, x + \delta) \cap H \in \mathcal{J}$. It can be proved analogously that $E(x) \cap H \backslash D_{\mathcal{J}}(A'_0)$ is of the $\mathcal{J}$-first category, $A'_0 \subset E(x) \cap H_0 \backslash \{x\}$ and for any $y \in A'_0$, $c > (b-a)/(x-y) > (f(x) - f(y))/(x-y)$ because $f(x) > b$, $f(y) < a$. Hence $\underline{f}'_{\mathcal{E}S}(x) = -\infty$.

Now we present the main result of this section.

Theorem 3.10. Under the same conditions on $(\mathbb{R}, \mathcal{J})$ and E as in Lemma 3.9, if $\underline{f}'_{\mathcal{E}S}(x) > -\infty$ $(\overline{f}'_{\mathcal{E}S}(x) < \infty)$ except for a set of the $\mathcal{J}$-first category, then f is $\mathcal{J}$-measurable.

Proof. This follows directly from Lemma 3.9.

Definition 3.11. A point $a \in \mathbb{R}^*$ is called a right sided essential derived number of $f : \mathbb{R} \longrightarrow \mathbb{R}$ at a point x if $\limsup_{h \to 0} |(x, x + h) \cap \{y : (f(x) - f(y))/(x-y) \in G\}|^*/h > 0$ for any $G \in \mathcal{O}^*$, $a \in G$. The set of all the right sided essential derived numbers of f at x will be denoted by $D^+_{ess}(f,x)$.

The upper (lower) right sided essential derivative of f at x is defined analogously as in Definition 2.2 and it is denoted by $\overline{f}^+_{ess}(x)$ $(\underline{f}^+_{ess}(x))$. The set $D^-_{ess}(f,x)$ of the left sided essential derived numbers of f at x and the extreme left sided essential derivatives $(\overline{f}^-_{ess})$ are defined analogously. (Some authors use $\overline{f}^+_{ap}$, $\underline{f}^+_{ap}$, $\overline{f}^-_{ap}$ ... (See [1].)).

Theorem 3.12. (See Mišík's question in [8].)

1. If $\overline{f}^+_{ess}(x) < \infty$ $(\underline{f}^+_{ess}(x) > -\infty)$ except for a set of Lebesgue measure zero, then f is Lebesgue measurable.

2. If $\overline{f}^-_{ess}(x) < \infty$ $(\underline{f}^-_{ess}(x) > -\infty)$ except for a set of Lebesgue measure zero, then f is Lebesgue measurable.

Proof. 1. Let $\mathcal{J} = \mathcal{N}$, $E^1(x) = [x,\infty)$ for each $x \in \mathbb{R}$. By Remark 3.3 we have $D(f,\mathcal{E}^1s,x) \subset D^+_{ess}(f,x)$, where $\mathcal{E}^1s = (\mathbb{R},\mathcal{N},E^1)$. Suppose that f is not Lebesgue measurable. By Lemma 3.9 there are sets S_1, S_2 such that $|S_1|^* > 0$, $|S_2|^* > 0$ and $\overline{f}'_{\mathcal{E}^1s}(x) = \infty$ for any $x \in S_1$ and $\underline{f}'_{\mathcal{E}^1s}(x) = -\infty$ for any $x \in S_2$. This is a contradiction.

2. If we let $E^2(x) = (-\infty,x]$ for each $x \in \mathbb{R}$, we obtain a contradiction to the assumption that the upper (lower) left sided essential derivative of f is almost everywhere less than ∞ (more than $-\infty$).

Theorem 3.13. Suppose the same conditions on $(\mathbb{R},\mathcal{J})$ and E as in Lemma 3.9. Let $\mathcal{E} = (\mathbb{R},\mathcal{J},E,\mathcal{C})$ be a generalized system of paths, where $\mathcal{C} \supset \mathcal{C}_2 = \{A \subset \mathbb{R} : A$ is of the $\mathcal{J}$-second category$\}$, $\phi \notin \mathcal{C}$. If $\overline{f}'_{\mathcal{E}}(x) < \infty$ $(\underline{f}'_{\mathcal{E}}(x) > -\infty)$ except for a set of the $\mathcal{J}$-first category, then f is $\mathcal{J}$-measurable.

Proof. Suppose that f is not $\mathcal{J}$-measurable. By Lemma 3.9 there are sets S_1, S_2 of the $\mathcal{J}$-second category such that $\overline{f}'_{\mathcal{E}s}(x) = \infty$ for any $x \in S_1$ and $\underline{f}'_{\mathcal{E}s}(x) = -\infty$ for any $x \in S_2$, where $\mathcal{E}s = (\mathbb{R},\mathcal{J},E)$. It is clear that $D(f,\mathcal{E},x) \supset D(f,\mathcal{E}s,x)$. Consequently $\overline{f}'_{\mathcal{E}}(x) = \infty$ for any $x \in S_1$ and $\underline{f}'_{\mathcal{E}}(x) = -\infty$ for any $x \in S_2$. This is a contradiction.

Theorem 3.14. Suppose the same conditions on $(\mathbb{R},\mathcal{J})$ and E as in Lemma 3.9. If $\overline{f}'_E(x) < \infty$ $(\underline{f}'_E(x) > -\infty)$ except for a set of the $\mathcal{J}$-first category, then f is $\mathcal{J}$-measurable.

Proof. Let $\mathcal{E}s = (\mathbb{R},\mathcal{J},E)$, $\mathcal{E}_{\mathcal{J}} = (\mathbb{R},\mathcal{J},E,2^{\mathbb{R}} \setminus \{\phi\})$, and $\mathcal{E} = (\mathbb{R},\mathcal{O},E,2^{\mathbb{R}} \setminus \{\phi\})$. Then $D(f,E,x) = D(f,\mathcal{E},x) \supset D(f,\mathcal{E}_{\mathcal{J}},x) \supset D(f,\mathcal{E}s,x)$. The assertion follows directly from Lemma 3.9.

Corollary 3.15. (See [2].) If one of the upper (lower) Dini derivatives of a function f is less than ∞ (more than $-\infty$), then f is $\mathcal{N}$-measurable.

4. Extreme E-derivatives

In this section we will develop a number of properties of extreme path derivatives and we will obtain a generalization of the results in [1] and [9].

Let E be a system of paths, i.e. a multifunction $E : \mathbb{R} \longrightarrow \mathbb{R}$ such that each $E(x)$ has x as a point of θ-accumulation. For any $n \in \mathbb{N}$ we define the relations E_n, E_n^-, E_n^+ as follows:

$$E_n(x) = E(x) \cap (x - 1/n, x + 1/n)$$

$$E_n^-(x) = E(x) \cap (x - 1/n, x)$$

$$E_n^+(x) = E(x) \cap (x, x + 1/n).$$

If $f : \mathbb{R} \longrightarrow \mathbb{R}$, define $f_0 : \mathbb{R}\times\mathbb{R} \setminus \Delta \longrightarrow \mathbb{R}$, by $f_0(x,y) = \frac{f(x)-f(y)}{x-y}$ where $\Delta = \{(x,x) \in \mathbb{R}\times\mathbb{R} : x \in \mathbb{R}\}$.

For $A \subset \mathbb{R}\times\mathbb{R}$ let $\mathrm{pr}A = \{x \in \mathbb{R} : \text{for some } y \in \mathbb{R}, (x,y) \in A\}$.

If $a = \pm\infty$, and if $r \in \mathbb{R}$ let $a \pm r = a$.

The set of all E-derived numbers of f at x is denoted by $D(f,E,x)$ (See Remark 2.10.) and the multifunction of E-derived numbers $D_{f,E} : \mathbb{R} \longrightarrow \mathbb{R}^*$ is defined as follows $D_{f,E}(x) = D(f,E,x)$. The proof of the next assertion is trivial and hence omitted.

Lemma 4.1. For any $a,b \in \mathbb{R}^*$, $a < b$ we have $D_{\bar{f},E}^{-}([a,b]) = \bigcap_{n=1}^{\infty} \mathrm{pr}(f_0^{-1}((a-1/n, b+1/n)) \cap \mathrm{Gr}E_n) = \bigcap_{n=1}^{\infty} \mathrm{pr}((f_0^{-1}((a-1/n,b+1/n)) \cap \mathrm{Gr}E_n^-) \cup (f_0^{-1}((a-1/n,b+1/n)) \cap \mathrm{Gr}E_n^+))$.

We turn now to a study of the Baire classification of $D_{f,E}$, $\bar{f}'_E$, $\underline{f}'_E$, f'_E.

Definition 4.2. Let A_α (M_α) denote the family of all sets of the Borel additive (multiplicative) class α. A multifunction $F : \mathbb{R} \longrightarrow \mathbb{R}^*$ is a lower (upper) semi Borel multifunction of the class α (briefly $F \in \ell B_\alpha$ $(F \in uB_\alpha)$) if $F^+((a,\infty]) \in A_\alpha (F^+([-\infty,a)) \in A_\alpha)$ for all $a \in \mathbb{R}$. Let $B_\alpha = \ell B_\alpha \cap uB_\alpha$. F is a Baire multifunction of the class α if $F \in B_\alpha$. For a single valued multifunction, it means for a function see [8].

Lemma 4.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. For $a \in \mathbb{R}$ let

$$S_a = \{(x,y) : f(x) - ax > f(y) - ay\}$$

$$T_a = \{(x,y) : f(x) - ax < f(y) - ay\}$$

(a) If $f \in \ell B_\alpha$, then $S_a = \bigcup_{i=1}^{\infty} A_i \times B_i$ where $A_i \in A_\alpha$, $B_i \in M_\alpha$

(b) If $f \in uB_\alpha$, then $T_a = \bigcup_{i=1}^{\infty} A_i \times B_i$ where $A_i \in A_\alpha$, $B_i \in M_\alpha$

Proof. (a) Let $r \in \mathbb{R}$. It is clear that $\{z \in \mathbb{R} : f(z) - az > r\} = \bigcup_{q \in \mathbb{Q}} \{z \in \mathbb{R} : f(z) > q\} \cap \{z \in \mathbb{R} : q > r + az\}$ where $\mathbb{Q} = \{a \in \mathbb{R} :$ a is a rational number$\}$. Since $\{z \in \mathbb{R} : f(z) > q\} \in A_\alpha$ and since $\{z \in \mathbb{R} : q > r + az\} \in A_0$, $\{z \in \mathbb{R} : f(z) - az > r\} \in A_\alpha$. The equality $S_a = \bigcup_{r \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) - ax > r\} \times \{y \in \mathbb{R} : f(y) - ay \leq r\}$ finishes the proof.

(b) This case is proved analogously, because $\{z \in \mathbb{R} : f(z) - az < r\} = \bigcup_{q \in \mathbb{Q}} \{z \in \mathbb{R} : f(z) < q\} \cap \{z \in \mathbb{R} : q < r + az\} \in A_\alpha$ and $T_a = \bigcup_{r \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) - ax < r\} \times \{y \in \mathbb{R} : f(y) - ay \geq r\}$.

Lemma 4.4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$. For any $a,b \in \mathbb{R}$, $a < b$

(a) $f_o^{-1}((a,\infty)) \cap GrE_n^- = S_a \cap GrE_n^-$

(b) $f_o^{-1}((-\infty,a)) \cap GrE_n^- = T_a \cap GrE_n^-$

(c) $f_o^{-1}((a,\infty)) \cap GrE_n^+ = T_a \cap GrE_n^+$

(d) $f_o^{-1}((-\infty,a)) \cap GrE_n^+ = S_a \cap GrE_n^+$

(e) $f_o^{-1}((a,b)) \cap GrE_n^- = S_a \cap T_b \cap GrE_n^-$

(f) $f_o^{-1}((a,b)) \cap GrE_n^+ = S_b \cap T_a \cap GrE_n^+$

where S_a, T_a are as in Lemma 4.3.

Proof. The trivial proof is omitted.

Lemma 4.5. If $GrE = \bigcup_{i=1}^{\infty} A_i \times B_i$ where $A_i \in A_\alpha$ and $B_i \subset \mathbb{R}$, then GrE_n, GrE_n^-, GrE_n^+ can be expressed as the union of a sequence of sets $X_i \times Y_i$, $X_i^- \times Y_i^-$, $X_i^+ \times Y_i^+$ where $X_i, X_i^-, X_i^+ \in A_\alpha$ and $Y_i, Y_i^-, Y_i^+ \subset \mathbb{R}$, $i = 1,2,3\ldots$, respectively.

Proof. We define the multifunctions $N, N^-, N^+ : \mathbb{R} \longrightarrow \mathbb{R}$ as $N(x) = (x-1/n, x+1/n)$, $N^-(x) = (x-1/n, x)$, $N^+(x) = (x, x+1/n)$ for all $x \in \mathbb{R}$. Then GrN, GrN^-, GrN^+ can be expressed as $\bigcup_{i=1}^{\infty} H_i \times G_i$, $\bigcup_{i=1}^{\infty} H_i^- \times G_i^-$, $\bigcup_{i=1}^{\infty} H_i^+ \times G_i^+$ where $H_i, G_i, H_i^-, G_i^-, H_i^+, G_i^+ \in A_0$ for any $i = 1,2,3\ldots$ respectively and the proof is finished by the equality $GrE_n = GrE \cap GrN$, $GrE_n^- = GrE \cap GrN^-$, $GrE_n^+ = GrE \cap GrN^+$, respectively.

Lemma 4.6. If $D_{\bar{f},E}([a,b]) \in M_{\alpha+1}$ for any $a,b \in \mathbb{R}^*$, $a < b$, then $\overline{f}'_E \in uB_{\alpha+1}$, $\underline{f}'_E \in \ell B_{\alpha+1}$, $D_{f,E} \in B_{\alpha+1}$, $f'_E \in B_{\alpha+1}$ (if f'_E exists).

Proof. Let $a \in \mathbb{R}$. Since $\overline{f}'^{-1}_E([a,\infty]) = D_{\bar{f},E}([a,\infty]) \in M_{\alpha+1}$, $\overline{f}'^{-1}_E([-\infty,a)) \in A_{\alpha+1}$. Analogously $\underline{f}'^{-1}_E([-\infty,a]) = D_{\bar{f},E}([-\infty,a]) \in M_{\alpha+1}$. Hence $\underline{f}'^{-1}_E((a,\infty]) \in A_{\alpha+1}$. $D_{\bar{\bar{f}},E}([-\infty,a)) = \mathbb{R} \setminus D_{\bar{f},E}([a,\infty]) \in A_{\alpha+1}$. Hence $D_{f,E} \in uB_{\alpha+1}$. $D_{\bar{\bar{f}},E}((b,\infty]) = \mathbb{R} \setminus D_{\bar{f},E}([-\infty,b]) \in A_{\alpha+1}$. Hence $D_{f,E} \in \ell B_{\alpha+1}$.

Theorem 4.7. Let $GrE = \bigcup_{i=1}^{\infty} A_i \times B_i$, $A_i \in A_\alpha$, $B_i \subset \mathbb{R}$. If $f \in B_\alpha$, then $D_{\bar{f},E}([a,b]) \in M_{\alpha+1}$ for any $a,b \in \mathbb{R}^*$, $a < b$ and by Lemma 4.6 $\overline{f}'_E \in uB_{\alpha+1}$, $\underline{f}'_E \in \ell B_{\alpha+1}$, $D_{f,E} \in B_{\alpha+1}$, $f'_E \in B_{\alpha+1}$ (if f'_E exists).

Proof. If $a = -\infty$, $b = \infty$, then $D_{\bar{f},E}([a,b]) = \mathbb{R} \in M_{\alpha+1}$. Let $a,b \in \mathbb{R}$, $a < b$. By Lemma 4.4e and Lemma 4.4f, respectively, and by Lemma 4.3 and Lemma 4.5, we have $f_o^{-1}((a-1/n, b+1/n)) \cap GrE_n^- = \bigcup_{i=1}^{\infty} X_i^n \times Y_i^n$ where $X_i^n \in A_\alpha$ and $f_o^{-1}((a-1/n, b+1/n)) \cap GrE_n^+ = \bigcup_{j=1}^{\infty} Z_j^n \times T_j^n$ where $Z_j^n \in A_\alpha$, respectively. Then by Lemma 4.1 we have $D_{\bar{f},E}([a,b]) = \bigcap_{n=1}^{\infty} (\bigcup_{i=1}^{\infty} X_i^n \cup \bigcup_{j=1}^{\infty} Z_j^n) \in M_{\alpha+1}$.

In the case $a \in \mathbb{R}$, $b = \infty$ ($a = -\infty$, $b \in \mathbb{R}$) we use analogously Lemma 4.4a,c (Lemma 4.4b,d).

For $E(x) = (x,\infty)$ and $E(x) = (-\infty,x)$, respectively we have

Corollary 4.8. (See [9].) The upper (lower) Dini derivatives of a Baire function of the class α are upper (lower) semiBorel functions of the class $\alpha+1$.

If E is unilateral, the following theorem is an improvement of Theorem 4.7.

Theorem 4.9. Let $GrE = \bigcup_{i=1}^{\infty} A_i \times B_i$, $A_i \in A_\alpha$, $B_i \subset \mathbb{R}$.

(a) If E is left sided on $\mathbb{R}$ and $f \in \ell B_\alpha$ $(f \in uB_\alpha)$, then $D_{\overline{f},E}([a,\infty]) \in M_{\alpha+1}$ $(D_{\overline{f},E}([-\infty,a])) \in M_{\alpha+1})$ for any $a \in \mathbb{R}$ and consequently $\overline{f}'_E \in uB_{\alpha+1}$, $D_{f,E} \in uB_{\alpha+1}$ $(\underline{f}'_E \in \ell B_{\alpha+1}$, $D_{f,E} \in \ell B_{\alpha+1})$.

(b) If E is right sided on $\mathbb{R}$ and $f \in \ell B_\alpha$ $(f \in uB_\alpha)$, then $D_{\overline{f},E}([-\infty,a]) \in M_{\alpha+1}$ $(D_{\overline{f},E}([a,\infty]) \in M_{\alpha+1})$ for any $a \in \mathbb{R}$ and consequently $\underline{f}'_E \in \ell B_{\alpha+1}$, $D_{f,E} \in \ell B_{\alpha+1}$ $(\overline{f}'_E \in uB_{\alpha+1}$, $D_{f,E} \in uB_{\alpha+1})$.

<u>Proof</u>. (a) In this case $f_o^{-1}((a-1/n,b+1/n)) \cap GrE_n^+ = \Phi$. Let $f \in \ell B_\alpha$ $(f \in uB_\alpha)$. By Lemma 4.4a (Lemma 4.4b), Lemma 4.3a (Lemma 4.3b), Lemma 4.5 and Lemma 4.1 we obtain $D_{\overline{f},E}([a,\infty]) \in M_{\alpha+1}$ $(D_{\overline{f},E}([-\infty,a]) \in M_{\alpha+1})$.

(b) In this case $f_o^{-1}((a-1/n,b+1/n)) \cap GrE_n^- = \Phi$. Let $f \in \ell B_\alpha$ $(f \in uB_\alpha)$. By Lemma 4.4d (Lemma 4.4c), Lemma 4.3a (Lemma 4.3b), Lemma 4.5 and Lemma 4.1 we obtain $D_{\overline{f},E}([-\infty,a]) \in M_{\alpha+1}$ $(D_{\overline{f},E}([a,\infty]) \in M_{\alpha+1})$.

The following corollary is an improvement of Corollary 4.8.

Corollary 4.10. (a) If $f \in \ell B_\alpha$, then $D^-f \in uB_{\alpha+1}$ and $D_+f \in \ell B_{\alpha+1}$.

(b) If $f \in uB_\alpha$, then $D_-f \in \ell B_{\alpha+1}$ and $D^+f \in uB_{\alpha+1}$ where D^-f, D^+f, D_-f, D_+f are upper left, upper right, lower left, lower right Dini derivative of f, respectively.

It is well known that the extreme path derivatives can behave badly. (For example, there is a continuous function F such that given any function f, a system of path E can be found such that $F'_E = f$.) In the following theorems we impose some restrictions on the system of paths as well as on the function. We shall show that under some conditions the E-derivative can have nice properties.

Theorem 4.11. (An improvement of Corollary 13 of [1].) If $f \in B_1$ and GrE is a F_σ set, then $D_{\bar{f},E}([a,b]) \in M_2$ for any $a,b \in \mathbb{R}^*$, $a < b$ and by Lemma 4.6 $\bar{f}'_E \in uB_2$, $\underline{f}'_E \in \ell B_2$, $D_{f,E} \in B_2$, $f'_E \in B_2$.

Proof. It is clear that if $A \subset \mathbb{R} \times \mathbb{R}$ is a bounded, closed set, then prA is closed. Since GrE is a F_σ set, GrE_n is a F_σ set for any $n = 1,2,\dots$. Since $f \in B_1$, f_0 is in Baire class one and $GrE_n \cap f_0^{-1}((a-1/n,b+1/n)) = \bigcup_{n=1}^{\infty} C_n$ where $C_n \subset \mathbb{R} \times \mathbb{R}$ and C_n is a closed set and without loss of generality we can suppose that C_n is bounded for any n. By Lemma 4.1 $D_{\bar{f},E}([a,b]) \in M_2$.

Definition 4.12. (See [10].) A multifunction $F : X \longrightarrow Y$ (X,Y are topological spaces) is said to be upper c-semicontinuous (ucsc) at $p \in X$ if for any open V containing $F(p)$ and such that $Y \setminus V$ is compact, there is a neighborhood U of p such that $F(x) \subset V$ for any $x \in U$. If F is ucsc at any $p \in X$, then F is said to be upper c-semicontinuous.

By Theorem 1 of [10], if $F : (\mathbb{R},\mathcal{O}) \longrightarrow (\mathbb{R},\mathcal{O})$ is a closed valued ucsc multifunction, then GrF is closed. Consequently, if F is usc, then GrF is closed.

Corollary 4.13. If E is a closed valued ucsc multifunction and $f \in B_1$, then $D_{\bar{f},E}([a,b]) \in M_2$ for any $a,b \in \mathbb{R}^*$, $a < b$ and by Lemma 4.6 $\bar{f}'_E \in uB_2$, $\underline{f}'_E \in \ell B_2$, $D_{f,E} \in B_2$, $f'_E \in B_2$.

The following theorem is the improvement of the main Theorem of [1] (Theorem 5 of [1]).

Theorem 4.14. If $f \in B_0$ and E is lsc, then $D_{\bar{f},E}([a,b]) \in M_1$ for any $a,b \in \mathbb{R}^*$, $a < b$ and by Lemma 4.6 $\bar{f}'_E \in uB_1$, $\underline{f}'_E \in \ell B_1$, $D_{f,E} \in B_1$, $f'_E \in B_1$.

Proof. It is clear that E_n is lsc for any n. We shall show that $A_n = pr(f_0^{-1}((a,b)) \cap GrE_n)$ is open. Let $x_0 \in A_n$. Then there is $y \in \mathbb{R}$ such that $(x_0,y) \in f_0^{-1}((a,b))$ and $y \in E_n(x_0)$. Since $f \in B_0$, f_0 is continuous. Hence there is $I \times J \ni (x_0,y)$ where $I,J \subset \mathbb{R}$ are open intervals such that $I \times J \subset f_0^{-1}((a,b))$. Since E_n is lsc at x_0, there is an open set $G \subset I$, $x_0 \in G$ such that $E_n(x) \cap J \neq \emptyset$ for any $x \in G$. Thus for any

$x \in G$ there is $y_x \in E_n(x) \cap J$. Hence $(x,y_x) \in GrE_n$. Since $(x,y_x) \in f_0^{-1}((a,b))$, $x \in A_n$ for any $x \in G$. By Lemma 4.1 $D_{\bar{f},E}([a,b]) \in M_1$.

Corollary 4.15. (See Corollary 10 of [1].) The congruent derivative and the extreme congruent derivative of a continuous function are in B_1 and B_2 respectively.

Proof. In this case $E(x) = E(0) + x$ for any $x \in \mathbb{R}$. Thus E is lsc.

Corollary 4.16. Let E be a system of paths that is bilateral, lsc and satisfies the intersection condition (See [3].). Let $f \in B_0$. If f'_E exists, then $f'_E \in B_1$ and f'_E has the Darboux property.

Proof. By Theorem 4.14, $f'_E \in B_1$ and by Theorem 6.4 of [3], f'_E has the Darboux property.

Definition 4.17. A multifunction $F : (\mathbb{R},\mathcal{T}) \longrightarrow (\mathbb{R}^*,\mathcal{O}^*)$ is said to be $\mathcal{T}$-measurable ($\mathcal{T}$-Borel measurable) if $F^-(G)$ has the $\mathcal{T}$-Baire property (is a $\mathcal{T}$-Borel set) for any $G \in \mathcal{O}^*$.

Theorem 4.18. If $f,E : (\mathbb{R},\mathcal{T}) \longrightarrow (\mathbb{R},\mathcal{O})$ are $\mathcal{T}$-measurable and $E(x)$ is $\mathcal{O}$-closed for each $x \in \mathbb{R}$, then $D_{f,E}$ is $\mathcal{T}$-measurable and consequently $\bar{f}'_E$, $\underline{f}'_E$, f'_E are $\mathcal{T}$-measurable.

Proof. By [6, p. 382], GrE is $(\mathcal{T}\times\mathcal{O})$-measurable. Hence GrE_n is $(\mathcal{T}\times\mathcal{O})$-measurable for any n. By [5, p. 62], if $A \subset \mathbb{R}\times\mathbb{R}$ is $(\mathcal{T}\times\mathcal{O})$-measurable, then prA has the $\mathcal{T}$-Baire property and by Lemma 4.1, $D_{f,E}$ is $\mathcal{T}$-measurable.

Corollary 4.19. (See [1] Theorem 16.) If f,E are $\mathcal{O}$-Borel measurable and $E(x)$ is $\mathcal{O}$-closed for any $x \in \mathbb{R}$, then $D_{f,E}$, $\bar{f}'_E$, $\underline{f}'_E$, f'_E are Lebesgue measurable and have the Baire property.

Theorem 4.20. Suppose the same conditions on $(\mathbb{R},\mathcal{T})$ and E as in Lemma 3.9 and E is $\mathcal{T}$-measurable and $E(x)$ is $\mathcal{O}$-closed for any $x \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an arbitrary function. If $\bar{f}'_E(x) < \infty$ ($\underline{f}'_E(x) > -\infty$) except for a set of the $\mathcal{T}$-first category, then f, $D_{f,E}$, $\bar{f}'_E$, $\underline{f}'_E$, f'_E are $\mathcal{T}$-measurable.

Proof. This follows from Theorem 3.14 and Theorem 4.18.

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