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ON A THEOREM OF BANACH CONCERNING PERIODIC FUNCTIONS

S. Banach (See [1].) proved that if $f : \mathbb{R} \to \mathbb{R}$ is a measurable periodic function with period 1, then $\limsup_n f(nx) = \operatorname*{ess\,sup}_{0 \le t \le 1} f(t)$ almost everywhere on $[0,1]$. In Theorem 1 we shall prove a generalization of Banach's theorem which yields a measure theoretic version (Proposition 1) and a category version (Proposition 2).

Let $\mathcal{S}$ be a σ-algebra of subsets of $\mathbb{R}$ and let $\mathcal{J} \subset \mathcal{S}$ be a proper σ-ideal of sets such that:

1° if $A \in \mathcal{J}$, then $A + a = \{x + a : x \in A\} \in \mathcal{J}$ for every $a \in \mathbb{R}$,

2° if $A \in \mathcal{J}$, then $a \cdot A = \{a \cdot x : x \in A\} \in \mathcal{J}$ for every $a \in \mathbb{R}$.

For every subset $E \subset \mathbb{R}$ put $E^* = \bigcup_{k=0}^{\infty} (E + k)$ and $E^{**} = \{x \in [0,1] : nx \in E^*$ for infinitely many $n \in \mathbb{N}\}$.

Definition 1. The pair $(\mathcal{S},\mathcal{J})$ has property $(**)$ means $[0,1] \setminus E^{**} \in \mathcal{J}$ for every $E \in \mathcal{S} \setminus \mathcal{J}$.

Let f be a $\mathcal{S}$-measurable function. Put $\mathcal{J}\text{-}\operatorname*{ess\,sup}_{0 \le t \le 1} f(t) = \inf\{g : \{ x : f(x) > g\} \in \mathcal{J}\}$.

Theorem 1. If the pair $(\mathcal{S},\mathcal{J})$ has property $(**)$ and if $f : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{S}$-measurable, periodic function with period 1, then $\{x : \limsup_n f(nx) \ne \mathcal{J}\text{-}\operatorname*{ess\,sup}_{0 \le t \le 1} f(t)\} \in \mathcal{J}$.

Proof. Suppose that $g = \mathcal{J}\text{-}\operatorname*{ess\,sup}_{0 \le t \le 1} f(t) < \infty$. Let $a < g$. Then $E_a = \{x \in [0,1] : f(x) > a\} \in \mathcal{S} \setminus \mathcal{J}$. From condition $(**)$ it follows that $[0,1] \setminus \{x \in [0,1] : nx \in E_a^*$ for infinitely many $n\} \in \mathcal{J}$. This means that $\limsup_n f(nx) \ge a$ except for a set belonging to $\mathcal{J}$.

Put $a_k = g - \frac{1}{k} < g$ for $k \in \mathbf{N}$. Then there exists a set $P_k \in \mathcal{J}$ such that $\limsup_n f(nx) \geq a_k$ for $x \notin P_k$. Let $P = \bigcup_{k=1}^{\infty} P_k$. Obviously $P \in \mathcal{J}$ and $\limsup_n f(nx) \geq a_k = g - \frac{1}{k}$ for $k \in \mathbf{N}$ and for $x \notin P$. Hence $\limsup_n f(nx) \geq g$ except on a set $P \in \mathcal{J}$.

Observe that $\limsup_n f(nx) \leq g$ except for a set belonging to $\mathcal{J}$, because if $\{x \in [0,1] : \limsup_n f(nx) > g\} \in \mathcal{S}\setminus\mathcal{J}$, then $\{x : f(x) > g\} \in \mathcal{S}\setminus\mathcal{J}$. But this contradicts the definition of $\mathcal{J}\text{-ess}\sup_{0\leq t\leq 1} f(t)$. Indeed, we have $\{x \in [0,1] : \limsup_n f(nx) > g\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in [0,1] : f(nx) > g\} \in \mathcal{S}\setminus\mathcal{J}$, so for every $m \in \mathbf{N}$ there exists $n_0 \geq m$ such that $\{x \in [0,1] : f(n_0x) > g\} \in \mathcal{S}\setminus\mathcal{J}$. Put $E_{n_0} = \{x \in [0,1] : f(n_0x) > g\}$. Then $n_0E_{n_0} = \{n_0x : x \in E_{n_0}\} \in \mathcal{S}\setminus\mathcal{J}$. But if $y \in n_0E_{n_0}$, then $f(y) > g$. Consequently, $\{x \in \mathbb{R} : f(x) > g\} \supset n_0E_{n_0}$ and $\{x \in \mathbb{R} : f(x) > g\} \in \mathcal{S}\setminus\mathcal{J}$.

Let $\mathcal{L}$ denote the σ-algebra of sets measurable in the sense of Lebesgue, let $\mathcal{N}$ be the σ-ideal of null sets and let μ be Lebesgue measure on $\mathbb{R}$.

Proposition 1. The pair $(\mathcal{L},\mathcal{N})$ has property (**).

Proof. Let $E \subset [0,1]$ be a measurable set of positive measure. Put $A_n = \{x \in [0,1] : nx \in E^*\}$ for every $n \in \mathbf{N}$. Then $E^{**} = \limsup_n A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$. It suffices to prove that $\mu(\bigcup_{n=m}^{\infty} A_n) = 1$ for every $m \in \mathbf{N}$. We have $A_n = \bigcup_{k=0}^{n-1} \frac{1}{n}(E + k)$. Observe that $\frac{1}{n}(E + k) \subset [\frac{k}{n}, \frac{k+1}{n}]$, so $\bigcup_{n=m}^{\infty} A_n$ has metric density greater than or equal to $\mu(E)$ on every interval $[\frac{k}{n},\frac{k+1}{n}]$, $n \geq m$, $0 \leq k \leq n - 1$.

Suppose that $\mu([0,1] \setminus \bigcup_{n=m}^{\infty} A_n) > 0$. Then there exists a point $x_0 \in [0,1]$, which is a density point of the set $[0,1] \setminus \bigcup_{n=m}^{\infty} A_n$. (We may assume that $x_0 \neq \frac{k}{n}$, $k \in \mathbf{N}$, $n \in \mathbf{N}$). Then there exists a sequence of intervals $\{[\frac{k_i}{n_i}, \frac{k_i+1}{n_i}]\}_{i\in\mathbf{N}}$ such that $x_0 \in [\frac{k_i}{n_i}, \frac{k_i+1}{n_i}]$, $n_i \to \infty$ and

$$\lim_{i\to\infty} \frac{\mu(([0,1] \setminus \bigcup_{n=m}^{\infty} A_n) \cap [\frac{k_i}{n_i}, \frac{k_i+1}{n_i}])}{\frac{1}{n_i}} = 1,$$

but this is impossible because

$$\lim_{i\to\infty} \frac{\mu(\bigcup_{n=m}^{\infty} A_n \cap [\frac{k_i}{n_i}, \frac{k_i+1}{n_i}])}{\frac{1}{n_i}} \geq \mu(E) > 0.$$

If $E \in \mathcal{L} - \mathcal{N}$ and $\mu(E \cap [0,1]) = 0$, then there exists $\ell \in \mathbb{N}$ such that $\mu(E \cap [\ell, \ell+1]) > 0$. The proof in this case is analogous and uses interval the form $[\frac{\ell+k}{n}, \frac{\ell+k+1}{n}]$ instead of $[\frac{k}{n}, \frac{k+1}{n}]$. Other changes are obvious.

Now, let $\mathcal{B}$ denote the σ-algebra of sets having the Baire property, i.e. $\mathcal{B} = \{G \triangle P;\ G$ is an open set and P is a meager set$\}$ and let $\mathcal{K}$ be the σ-ideal of meager sets.

Proposition 2. The pair $(\mathcal{B}, \mathcal{K})$ has property $(**)$.

Proof. Let $E \in \mathcal{B}\setminus\mathcal{K}$, $E \subset [0,1]$. Again put $A_n = \{x \in [0,1] : nx \in E^*\} = \bigcup_{k=0}^{n-1} \frac{1}{n}(E + k)$, for every $n \in \mathbb{N}$. Then $E^{**} = \limsup_n A_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} A_n$. It suffices to prove that $\bigcup_{n=m}^{\infty} A_n$ is a residual set for $m \in \mathbb{N}$. Suppose to the contrary that there exists m such that $\bigcup_{n=m}^{\infty} A_n$ is not a residual set. Obviously, $\frac{1}{n}(E + k) \subset [\frac{k}{n}, \frac{k+1}{n}]$ for $k = 0,\ldots,n-1$ and $\frac{1}{n}(E + k)$ is a set of the second category. So $\bigcup_{n=m}^{\infty} A_n$ has the Baire property and is of the second category on every interval $[\frac{k}{n}, \frac{k+1}{n}]$ for $k = 0,\ldots,n-1$ and $n \geq m$. We have $\bigcup_{n=m}^{\infty} A_n = (G - P_1) \cup P_2$, where G is an open set and P_1, P_2 are of the first category. Then $[0,1]\setminus \bigcup_{n=m}^{\infty} A_n = [([0,1]\setminus G) \cup P_1] \cap [[0,1]\setminus P_2] \notin \mathcal{N}$. So $[0,1]\setminus G$ is a closed set of the second category. Hence, there exists an interval $[a,b]$, $(a < b)$ such that $[a,b] \subset [0,1]\setminus G$. There exist $n \geq m$ and $k \in \mathbb{N}$ such that $[\frac{k}{n}, \frac{k+1}{n}] \subset [a,b]$. Then $\bigcup_{n=m}^{\infty} A_n \cap [\frac{k}{n}, \frac{k+1}{n}] \in \mathcal{S}$ because $G \cap [\frac{k}{n}, \frac{k+1}{n}] = \emptyset$. But this is a contradiction.

In the general case (i.e. $E - [0,1] \notin \mathcal{S}$ and $E \cap [0,1] \in \mathcal{S}$) the proof is analogous.

Observe that the pair $(\mathcal{L} \cap \mathcal{B}, \mathcal{N} \cap \mathcal{K})$ does not have property $(**)$. Let $E \subset [0,1]$ be a meager set of positive measure. Then $E \in \mathcal{L} \cap \mathcal{B}\setminus\mathcal{N} \cap \mathcal{K}$. But $E^{**} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=0}^{n-1} \frac{1}{n} (E + k)$ is a meager set, so $[0,1]\setminus E^{**} \notin \mathcal{N} \cap \mathcal{K}$.

Now we shall consider the set $\{x \in [0,1]\colon \limsup_n f(nx) \neq \mathcal{S}\text{-ess}\sup_{0\leq t\leq 1} f(t)\}$.

Let $A \subset [0,1)$. Put

$$U(A) = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [\frac{1}{n} (A \cup (A + 1) \cup \ldots \cup (A + n - 1))] =$$

$$= \limsup_n [\frac{1}{n} (A \cup (A + 1) \cup \ldots \cup (A + n - 1))].$$

It is easy to see that $U(A \cup B) = U(A) \cup U(B)$ for all $A,B \subset [0,1]$ and if $A \subset B$, then $U(A) \subset U(B)$. If $A_n \subset A$ for every n, then $\bigcup_{n=1}^{\infty} U(A_n) \subset U(A)$.

Let x be an irrational number and put $A = \{x\}$. We shall show that $U(A) = \emptyset$. Suppose otherwise that there exists a real number z such that $z \in U(A) = \limsup_n \frac{1}{n} (A \cup (A + 1) \cup \ldots \cup (A + n - 1))$. Then there exists a sequence $\{n_m\}$ tending to infinity such that

$$z \in \bigcap_{m=1}^{\infty} \frac{1}{n_m} (A \cup (A + 1) \cup \ldots \cup (A + n_m - 1)).$$

Hence there exist sequences $\{n_m\}$ and $\{k_m\}$, $k_m \leq n_m - 1$ such that $z = (x + k_m)/n_m$ for every $m \in \mathbb{N}$. Let $n_{m_1} \neq n_{m_2}$. We have

$$(x + k_{m_1})/n_{m_1} = (x + k_{m_2})/n_{m_2}$$

so $x = (k_{m_2} n_{m_1} - k_{m_1} n_{m_2})/(n_{m_2} - n_{m_1})$. But this is impossible because x is an irrational number. Consequently, $U(A) = \emptyset$. It is also easy to see that if x is an irrational number, then all numbers of the form $\frac{x+k}{n}$, $k \leq n$, are different.

If $A = \bigcup_{n=1}^{\infty} A_n$, then $\bigcup_{n=1}^{\infty} U(A_n) \subset U(\bigcup_{n=1}^{\infty} A_n) = U(A)$ because of the monotonicity of U. Observe that the reverse inclusion need not hold. We shall find a sequence of sets $\{A_n\}$ such that $U(\bigcup_{n=1}^{\infty} A_n) \neq \bigcup_{n=1}^{\infty} U(A_n)$.

Let z be an irrational number from the interval $[0,1]$. Put $x_n = n \cdot z - [nz]$, $A_n = \{x_n\}$ for $n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\infty} A_n$. Obviously, x_n is an irrational number; so $U(A_n) = \emptyset$ for $n \in \mathbb{N}$. We shall show that $z \in U(A)$. For every $n \in \mathbb{N}$ we have

$$z = \frac{x_n + [nz]}{n}$$

and $[nz] < n$, because $z < 1$. Consequently, $z \in \bigcap_{n=1}^{\infty} \frac{1}{n}(A \cup (A + 1) \cup \ldots \cup \ldots \cup (A + n - 1)) \subset U(A)$.

Observe that if $x = \frac{p_1}{q_1}$ (irreducible fraction), then $\frac{p}{q} \in U(\{x\})$ if and only if q is a multiple of q_1. Conversely, for all natural numbers $r,s \in \mathbb{N}$ such that $r < sq_1$ and $r \nmid sq_1$ we have $\frac{r}{sq_1} \in U(\{\frac{p_1}{q_1}\})$.

Theorem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{I}$-measurable, periodic function with period 1 and $M = \mathcal{I}\text{-}\operatorname{ess\,sup}_{0 \leq t \leq 1} f(t)$. Then

$$\{x \in [0,1) : \limsup_n f(nx) > M\} = \bigcup_{k=1}^{\infty} U(f^{-1}((M + \tfrac{1}{k}, \infty)) \cap [0,1)).$$

Proof. Put $A = \{x \in [0,1) : \limsup_n f(nx) > M\}$ and let $x \in A$. Then $x \in [0,1)$ and $\limsup_n f(nx) > M$. There exists $k \in \mathbb{N}$ such that $\limsup_n f(nx) > M + \frac{1}{k}$. So there exists a sequence $\{n_m\}$ tending to infinity such that $f(n_m x) > M + \frac{1}{k}$ for every $m \in \mathbb{N}$. Observe that $x \in U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$. Indeed, let $f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1) = B$.

From the assumption that f is a periodic function, it follows that $f(n_m x) > M + \frac{1}{k}$ if and only if $n_m x \in B \cup (B+1) \cup \ldots \cup (B + n_m - 1)$, i.e. $x \in \frac{1}{n_m}(B \cup (B+1) \cup \ldots \cup (B + n_m - 1))$. We have $f(n_m x) > M + \frac{1}{k}$ for every $m \in \mathbb{N}$. So $x \in \bigcap_{m=1}^{\infty} \frac{1}{n_m}(B \cup (B+1) \cup \ldots \cup (B + n_m - 1)) \subset$ $\subset \limsup_n (B \cup (B+1) \cup \ldots \cup (B + n - 1) = U(B)$. This means that $A \subset \bigcup_{k=1}^{\infty} U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$.

Conversely, let $x \in \bigcup_{k=1}^{\infty} U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$. There exists $k \in \mathbb{N}$ such that $x \in U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$. Put $f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1) = B$. We have $x \in U(B)$. So there exists a sequence $\{n_m\}$ tending to infinity such that $x \in \bigcap_{m=1}^{\infty} \frac{1}{n_m}(B \cup (B+1) \cup \ldots \cup (B + n_m - 1))$. This means that $n_m x \in$ $\in B \cup (B+1) \cup \ldots \cup (B + n_m - 1)$ for every $m \in \mathbb{N}$. Hence $f(n_m x) > M + \frac{1}{k}$ (because f is a periodic function with period 1 and $B + \ell = f^{-1}((M + \frac{1}{k}, \infty)) \cap$ $\cap [\ell, \ell + 1)$ for every $\ell \in \mathbb{N}$). Consequently, $\limsup_n f(nx) \geq \limsup_m f(n_m x) \geq$ $\geq M + \frac{1}{k} > M$ and $x \in A$.

The converse theorem also holds:

Theorem 3. If $B \in \mathcal{S}$ and if $B = \bigcup_{k=1}^{\infty} U(A_k)$ for some increasing sequence $\{A_k\}$ of $\mathcal{S}$-measurable subsets of $[0,1)$, then there exists a $\mathcal{S}$-measurable periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 1 such that

$$\{x \in [0,1) : \limsup_n f(nx) > \mathcal{S}\text{-}\operatorname*{ess\,sup}_{0 \leq t \leq 1} f(t)\} = B.$$

Proof. Put

$$f(x) = \begin{cases} 1/\min\ \{\ell : x \in A_\ell\} & \text{if } x \in \bigcup_{\ell=1}^{\infty} A_\ell \\ 0 & \text{if } x \in [0,1) \setminus \bigcup_{\ell=1}^{\infty} A_\ell . \end{cases}$$

We shall consider the periodic extension of f on $\mathbb{R}$. Obviously $\mathcal{S}\text{-}\operatorname*{ess\,sup}_{0 \leq t \leq 1} f(t) = 0$.

Observe that $\{x \in [0,1) : \limsup_n f(nx) > 0\} = B$.

Let $x \in B$. Evidently, $x \in [0,1)$. There exists $k \in \mathbb{N}$ such that $x \in U(A_k) = \limsup_n \frac{1}{n} (A_k \cup (A_k + 1) \cup \dots \cup (A_k + n - 1))$. So there exist $k \in \mathbb{N}$ and a sequence $\{n_m\}$ tending to infinity such that $n_m x \in A_k \cup (A_k + 1) \cup \dots \cup (A_k + n_m - 1)$ for every $m \in \mathbb{N}$. From the periodicity of f we have

$$f(n_m x) = 1/\min \{\ell : n_m x \in A_\ell \cup (A_\ell + 1) \cup \dots \cup (A_\ell + n_m - 1)\} \geq 1/k$$

for every $m \in \mathbb{N}$. Hence $\limsup_n f(nx) \geq \limsup_m f(n_m x) \geq 1/k$. Therefore $x \in \{y \in [0,1) : \limsup_n f(ny) > 0\}$. Consequently, $B \subset \{x \in [0,1) : \limsup_n f(nx) > 0\}$.

Let $x \in \{y \in [0,1) : \limsup_n f(ny) > 0\}$. Then $\limsup_n f(nx) > 0$. Hence there exist $k \in \mathbb{N}$ and a sequence $\{n_m\}$ tending to infinity such that $f(n_m x) > 1/k$ for every $m \in \mathbb{N}$. We have

$$f(n_m x) = 1/\min \{\ell : n_m x - [n_m x] \in A_\ell\} > 1/k$$

for every $m \in \mathbb{N}$, and $\{A_\ell\}$ is an increasing sequence of sets; so $n_m x \in A_k + [n_m x]$. There are two cases:

1° $x = 0$. Then the sequence $\{f(nx)\}$ is constant and equals $f(0)$. Thus, $\limsup_n f(nx) = f(0) > 1/k$. Hence $0 \in A_k$, $0 \in \frac{1}{2}(A_k \cup (A_k + 1)), \dots$, $0 \in \frac{1}{n} (A_k \cup (A_k + 1) \cup \dots \cup (A_k + n - 1)), \dots$. Consequently, $0 \in \bigcap_{n=1}^{\infty} (1/n)(A_k \cup (A_k + 1) \cup \dots \cup (A_k + n - 1)) \subset U(A_k) \subset B$.

2° $0 < x < 1$. Then $n_m x < n_m$ and $[n_m x] \leq n_m - 1$ for every $m \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and a sequence $\{n_m\}$ tending to infinity such that $n_m x \in A_k \cup (A_k + 1) \cup \dots \cup (A_k + n_m - 1)$ for $m \in \mathbb{N}$. Hence $x \in \bigcap_{m=1}^{\infty} (1/n_m)(A_k \cup (A_k + 1) \cup \dots \cup (A_k + n_m - 1)) \subset$ $\subset \limsup_n \frac{1}{n} (A_k \cup (A_k + 1) \cup \dots \cup (A_k + n - 1)) = U(A_k) \subset B$.

Consequently $\{x \in [0,1) : \limsup_n f(nx) > 0\} \subset B$.

Let $A \subset [0,1)$. But

$$\mathcal{L}(A) = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m}^{\infty} [\frac{1}{n}(A \cup (A+1) \cup \ldots \cup (A+n-1))] =$$

$$= \liminf_n [\frac{1}{n}(A \cup (A+1) \cup \ldots \cup (A+n-1))].$$

It is easy to see that $\mathcal{L}(A \cap B) = \mathcal{L}(A) \cap \mathcal{L}(B)$ for all $A, B \subset [0,1)$ and if $A \subset B$, then $\mathcal{L}(A) \subset \mathcal{L}(B)$. If $A = \bigcup_{n=1}^{\infty} A_n$, then $\bigcup_{n=1}^{\infty} \mathcal{L}(A_n) \subset \mathcal{L}(\bigcup_{n=1}^{\infty} A_n) = \mathcal{L}(A)$ by the monotonicity of $\mathcal{L}$. The same example as for the operation U shows that the reverse inclusion need not hold.

Theorem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{J}$-measurable periodic function with period 1 and $M = \mathcal{J}\text{-}\operatorname{ess\,sup}_{0 \leq t \leq 1} f(t)$. Then

$$\{x \in [0,1) : \limsup_n f(nx) < M\} = \bigcup_{k=1}^{\infty} \mathcal{L}(f^{-1}((-\infty, M - \tfrac{1}{k})) \cap [0,1)).$$

Proof. Put $A = \{x \in [0,1) : \limsup_n f(nx) < M\}$ and let $x \in A$. Then $x \in [0,1)$ and $\limsup_n f(nx) < M$. There exists $k \in \mathbb{N}$ such that $\limsup_n f(nx) < M - \frac{1}{k}$. So there exists $n_0 \in \mathbb{N}$ such that $f(nx) < M - \frac{1}{k}$ for $n \geq n_0$. Observe that $x \in \mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. Indeed, let $f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1) = B$. From the assumption that f is a periodic function, it follows that $f(nx) < M - \frac{1}{k}$ if and only if $nx \in B \cup (B+1) \cup \ldots \cup (B+n-1)$, i.e. $x \in \frac{1}{n}(B \cup (B+1) \cup \ldots \cup (B+n-1))$. We have $f(nx) < M - \frac{1}{k}$ for $n \geq n_0$. Hence $x \in \liminf_n \frac{1}{n}(B \cup (B+1) \cup \ldots \cup (B+n-1)) = \mathcal{L}(B)$. This means that $A \subset \bigcup_{k=1}^{\infty} \mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$.

Conversely, let $x \in \bigcup_{k=1}^{\infty} \mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. There exists $k \in \mathbb{N}$ such that $x \in \mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. Put $f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1) = B$. We have $x \in \mathcal{L}(B)$. Thus there exists $n_0 \in \mathbb{N}$ such that $x \in \frac{1}{n}(B \cup (B+1) \cup \ldots \cup (B+n-1))$ for $n \geq n_0$. This means that $nx \in (B \cup (B+1) \cup \ldots \cup (B+n-1))$ for $n \geq n_0$. Therefore $f(nx) < M - \frac{1}{k}$ (because f is a periodic function with period 1 and $B + \ell = f^{-1}((-\infty, M - \frac{1}{k})) \cap [\ell, \ell + 1)$ for every $\ell \in \mathbb{N}$). Consequently, $\limsup_n f(nx) \leq M - \frac{1}{k} < M$ and $x \in A$.

The converse theorem also holds.

Theorem 5. If $B \in \mathcal{S}$ and if $B = \bigcup_{k=1}^{\infty} \mathcal{L}(A_k)$ for some increasing sequence $\{A_k\}$ of $\mathcal{S}$-measurable subsets of $[0,1)$, then there exists a $\mathcal{S}$-measurable periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 1 such that

$$\{x \in [0,1) : \limsup_n f(nx) < \mathcal{S}\text{-}\operatorname*{ess\,sup}_{0 \leq t \leq 1} f(t)\} = B.$$

Proof. Put

$$f(x) = \begin{cases} (-1)/\min\{\ell : x \in A_\ell\} & \text{if } x \in \bigcup_{\ell=1}^{\infty} A_\ell \\ 0 & \text{if } x \in [0,1) - \bigcup_{\ell=1}^{\infty} A_\ell . \end{cases}$$

We shall consider the periodic extension of f on $\mathbb{R}$. Obviously, $\mathcal{S}\text{-}\operatorname*{ess\,sup}_{0 \leq t \leq 1} f(t) = 0$.

Observe that $\{x \in [0,1) : \limsup_n f(nx) < 0\} = B$.

Let $x \in B$. Evidently, $x \in [0,1)$. There exists $k \in \mathbb{N}$ such that $x \in \mathcal{L}(A_k) = \liminf_n \frac{1}{n} (A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1))$. So there exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $nx \in (A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1))$ for $n \geq n_0$. From the periodicity of f we have

$$f(nx) = (-1)/\min\{\ell : nx \in A_\ell \cup (A_\ell + 1) \cup \ldots \cup (A_\ell + n - 1)\} \leq (-1)/k$$

for $n \geq n_0$. Hence $\limsup_n f(nx) \leq (-1)/k < 0$. Therefore $x \in \{y \in [0,1) : \limsup_n f(ny) < 0\}$. Consequently, $B \subset \{x \in [0,1) : \limsup_n f(nx) < 0\}$.

Let $x \in \{y \in [0,1) : \limsup_n f(ny) < 0\}$. Then $\limsup_n f(nx) < 0$. Hence there exist $k \in \mathbb{N}$ and $n_0 \in N$ such that $f(nx) < (-1)/k$ for $n \geq n_0$. We have $f(nx) = (-1)/\min\{\ell : nx - [nx] \in A_\ell\} < (-1)/k$ for $n \geq n_0$ and $\{A_\ell\}$ is an increasing sequence of sets. Hence $nx \in A_k + [nx]$ for $n \geq n_0$. There are two cases:

1° $x = 0$. Then the sequence $\{f(nx)\}$ is constant and equals $f(0)$. Thus $\limsup_n f(nx) = f(0) < (-1)/k$. Hence $0 \in A_k$, $0 \in \frac{1}{2}(A_k \cup (A_k + 1)), \ldots, 0 \in \frac{1}{n}(A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1)), \ldots$. Consequently, $0 \in \bigcap_{n=1}^{\infty} (1/n)(A_k \cup (A + 1) \cup \ldots \cup (A_k + n - 1)) \subset \mathcal{L}(A_k) \subset B$.

2° $0 < x < 1$. Then $nx < n$ and $[nx] \leq n - 1$ for every $n \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $nx \in (A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1))$ for $n \geq n_0$. Hence $x \in \bigcap_{n=n_0}^{\infty} (1/n)(A_k \cup (A_k + 1)) \cup \ldots \cup (A_k + n - 1)) \subset \liminf_n \frac{1}{n}(A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1)) = \mathcal{L}(A_k) \subset B$.

Consequently, $\{x \in [0,1) : \limsup_n f(nx) < 0\} \subset B$.

References

[1] Scottish book, Solution by S. Banach of Problem 162 of H. Steinhaus.

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