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ON A THEOREM OF BANACH CONCERNING PERIODIC FUNCTIONS

S. Banach (See [1].) proved that if $f : \mathbb{R} \to \mathbb{R}$ is a measurable periodic function with period 1, then $\lim_{n \to \infty} \sup_{0 \le t \le 1} f(x) = \lim_{n \to \infty} \lim_{n \to \infty} \inf_{0 \le t \le 1} f(x) = \lim_{n \to \infty} \lim_{n \to \infty$ on $[0,1]$. In Theorem 1 we shall prove a generalization of Banach's theorem which yields a measure theoretic version (Proposition 1) and a category version (Proposition 2).

Let \downarrow be a σ -algebra of subsets of R and let $\downarrow \circ \downarrow$ be a proper σ -ideal of sets such that:

1^{*} if $A \in \mathcal{S}$, then $A + a = \{x + a : x \in A\} \in \mathcal{S}$ for every $a \in \mathbb{R}$, 2^{*} if $A \in \mathcal{A}$, then $a \cdot A = \{a \cdot x : x \in A\} \in \mathcal{A}$ for every $a \in \mathbb{R}$. 00 For every subset $E \subseteq R$ put $E^* = U$ $(E + k)$ and $E^{**} = \{x \in [0,1] :$ $k=0$ $nx \in E^*$ for infinitely many $n \in N$.

Definition 1. The pair $(4,4)$ has property $(\ast \ast)$ means $[0,1] \setminus$ $E^{**} \in 3$ for every $E \in 4 \setminus 3$.

Let f be a $\texttt{A-measurable}$ function. Put $\texttt{A-res}$ sup $f(t)$ = Oátál = $\inf\{g : \{ x : f(x) > g \} \in \mathcal{A} \}.$

Theorem 1. If the pair $(4, 4)$ has property $(\ast \ast)$ and if $f : \mathbb{R} \to \mathbb{R}$ is a $\texttt{A-measurable},$ periodic function with period 1, then $\{x : \text{lim sup } f(nx) \}$ \neq n 4 -ess sup $f(t)$ } ϵ 4 . Oátál

Proof. Suppose that $g = 1$ -ess sup $f(t) < \infty$. Let $a < g$. Then $E_a =$ Oátál = $\{x \in [0,1] : f(x) > a\} \in \mathcal{L}\backslash\mathcal{A}$. From condition $(**)$ it follows that $[0,1] \setminus \mathcal{A}$ $\{x \in [0,1] : nx \in E_A^* \text{ for infinitely many } n\} \in \mathcal{A}.$ This means that lim sup f(nx) \ge a except for a set belonging to \downarrow .

Put $a_k = g - \frac{1}{k}$ $\langle g \rangle$ for $k \in \mathbb{N}$. Then there exists a set $P_k \in \mathbb{R}$ superior that that lim sup $f(nx)$ a_k for $x \neq P_k$. Let $P = \bigcup_{k=1}^{n} P_k$. Obviously $P \in \mathcal{A}$ $a_k = g - \frac{1}{k} \langle g \text{ for } k \in \mathbb{N}.$ Then there exists a set $P_k \in \mathbb{R}$ is

m sup $f(nx) \ge a_k$ for $x \ne P_k$. Let $P = \bigcup_{k=1}^{\infty} P_k$. Obviously P

sup $f(nx) \ge a_k = g - \frac{1}{k}$ for $k \in \mathbb{N}$ and for $x \ne P$. Hence $\lim_{n \to \infty} \frac{1}{n$ and $\limsup_{n \to \infty} f(nx)$ $\geq a_k = g - \frac{1}{k}$ for $k \in \mathbb{N}$ and for $x \neq P$. Hence $\limsup_{n \to \infty} f(n)$ $f(nx) \ge g$ except on a set $P \in \mathcal{A}$.

Observe that $\lim_{n \to \infty} \sup_{n \to \infty} f(nx)$ is generate to a set belonging to s, because if $\{x \in [0,1]: \limsup_{n} f(nx) > g\} \in \mathcal{L}\backslash\mathcal{L}, \text{ then } \{x : f(x) > g\} \in \mathcal{L}\backslash\mathcal{L}. \text{ But}$ this contradicts the definition of \Box -ess sup $f(t)$. Indeed, we have o o U<mark>≦</mark>tál { $x \in [0,1]$: lim sup $f(nx) > g$ } = $\frac{0}{m-1}$ $\frac{0}{n-m}$ { $x \in [0,1]$: $f(nx) > g$ } \in
so for every $m \in \mathbb{N}$ there exists $n_0 \ge m$ such that { $x \in [0,1]$: $f(n_0x)$
 $\in \mathcal{L}\backslash\mathcal{S}$. Put $E_{n_0} = \{x \in [0,1] : f(n_0x) > g\}$. so for every $m \in N$ there exists $n_0 \ge m$ such that $\{x \in [0,1] : f(n_0x) > g\}$ ϵ **4** 4. Put $E_{n_0} = \{x \in [0,1] : f(n_0x) > g\}$. Then $n_0E_{n_0} = \{n_0x : x \in E_{n_0}\}\epsilon$ $f(nx) \rightarrow g$ except on a set $P \in \mathcal{A}$.

(bserve that $\limsup_{n} f(nx) \rightarrow g$) $\in \mathcal{A}\setminus\mathcal{A}$, then $\{x : f(x) > g\} \in \mathcal{A}\setminus\mathcal{A}$. But

this contradicts the definition of $\mathcal{A}-e^{-2}$ and $f(t)$. Indeed, we have
 $\{x \in [0,1] : \limsup_{$ Observe that $\lim_{n} \sup_{n} f(nx) \leq g$ except for a set belonging to 2, because

if $\{x \in [0,1] : \lim_{n} \sup_{n} f(nx) > g\} \in \mathcal{A}\backslash \mathcal{A}$, then $\{x : f(x) > g\} \in \mathcal{A}\backslash \mathcal{A}$. But

this contradicts the definition of 3-ess sup $f(t)$. In

Let $\boldsymbol{\ell}$ denote the σ -algebra of sets measurable in the sense of Lebesgue, let n be the σ -ideal of null sets and let μ be Lebesgue measure on $\mathbb R$.

Proposition 1. The pair (x, π) has property $(**)$.

Proof. Let $E \subset [0,1]$ be a measurable set of positive measure. Put $A_n = \{x \in [0,1] : nx \in E^* \}$ for every $n \in \mathbb{N}$. Then $E^{**} = \limsup_{n \to \infty} A_n =$ **Proof.** Let $E \subseteq [0,1]$ be a measurable set of positive measure. Put $A_n = \{x \in [0,1] : nx \in E^*\}$ for every $n \in N$. Then $E^{**} = \lim_{n \to \infty} \sup_{n \to \infty} A_n = \lim_{n \to \infty} A_n$. It suffices to prove that $\mu(\begin{array}{cc} 0 & A_n \end{array}) = 1$ for every $\begin{array}{cccc}\n\text{...} & \text{...} & \text{...} \\
\text{...} & \text{...} & \text{...} \\
\end{array}$ It suffices to prove that μ (μ A_n) = 1 for

n⁻¹ $\frac{1}{n}$ (E + k). Observe that $\frac{1}{n}$ (E + k) $\frac{k+1}{n}$ We have $A_n = \bigcup_{k=0}^{\infty} \frac{1}{n} (E + k)$. Observe that $\frac{1}{n} (E + k) \in [\frac{n}{n}, \frac{n}{n}]$ $\frac{1}{k=0}$ $\frac{1}{n}$ (E + k). Observe that $\frac{1}{n}$ (E + k) \in $\left[\frac{k}{n}\right]$, $\frac{k+1}{n}$, so $\frac{0}{n=m}$ <u>k k+1</u> has metric density greater than or equal to $\mu(E)$ on every interval $[\frac{1}{n}, \frac{1}{n}]$, $n \geq m$, $0 \leq k \leq n - 1$.

Suppose that $\mu([0,1] \setminus \bigcup_{n=m}^{\infty} A_n) > 0$. Then there exists a point $\begin{array}{ccc} \texttt{...} \ \texttt$ x_{o} ϵ [0,1], which is a density point of the set $\text{[0,1]} \setminus \text{U}$ A_n. (We may assume that $x_0 \neq \frac{k}{n}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. Then there exists a sequence of assume that $x_0 \neq \frac{1}{n}$, $k \in \mathbb{N}$, $n \in \mathbb{N}$. Then there
intervals $\{[\frac{k_1}{n_1}, \frac{k_1+1}{n_1}]\}_{i \in \mathbb{N}}$ such that $x_0 \in [\frac{k_1}{n_1}, \frac{k_1+1}{n_1}]$

$$
\lim_{i \to \infty} \frac{\mu(((0,1] \setminus \bigcup_{n=m}^{\infty} A_n) \cap [\frac{k_i}{n_i}, \frac{k_i+1}{n_i}])}{\frac{1}{n_i}} = 1,
$$

but this is impossible because

$$
\lim_{i \to \infty} \frac{\mu(\begin{array}{c} \omega \\ \omega \\ n=m \end{array} A_n \cap [\frac{k_i}{n_i}, \frac{k_i+1}{n_i}])}{\frac{1}{n_i}} \Rightarrow \mu(E) > 0.
$$

If $E \in 4 - 9$ and $\mu(E \cap [0,1]) = 0$, then there exists $\ell \in N$ such that $\mu(E \cap [\ell, \ell+1]) > 0$. The proof in this case is analogous and uses interval the form $\left[\frac{s+k}{n}, \frac{s+k+1}{n}\right]$ instead of $\left[\frac{k}{n}, \frac{k+1}{n}\right]$. Other changes are obvious. $\frac{1}{n}$, $\frac{\ell+k+1}{n}$ instead of $\left[\frac{k}{n}, \frac{k+n}{n}\right]$

Now, let B denote the σ -algebra of sets having the Baire property, i.e. $B = \{G \triangle P; G \text{ is an open set and } P \text{ is a meager set}\}$ and let K be the σ -ideal of meager sets.

Proposition 2. The pair (B,K) has property $(**)$.

Proposition 2. The pair (\mathbf{B}, \mathbf{K}) has property $(\ast \ast)$.

Proof. Let $\mathbf{E} \in \mathbf{B} \setminus \mathbf{K}$, $\mathbf{E} \in [0,1]$. Again put $\mathbf{A_n} = \{ \mathbf{x} \in [0,1] : \mathbf{nx} \in \mathbf{E}^* \} = \frac{n-1}{\mathbf{x}} \mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} \cdot \mathbf{B} \cdot$ **Proof.** Let $E \in B \setminus K$, $E \subseteq [0,1]$. Again put $A_n = \{x \in [0,1] : nx \in E^*\} =$ $\begin{array}{ccccccccc}\nn-1 & & & & & & \infty & \infty & \mathbb{Z} & \mathbb{Z$ It suffices to prove that $\begin{array}{ccc} \mathsf{U} & \mathsf{A}_\mathbf{n} & \text{is a residual set for }~\mathfrak{m}\in\mathsf{N}. \end{array}$ Suppose to the contrary that there exists **m** such that $\frac{0}{n-m}$ **A**_n is not
Obviously, $\frac{1}{n}$ (E + k) c $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ for $k = 0, ..., n - 1$ and $\frac{1}{n}$ 00 the contrary that there exists **m** such that U A_n is not a residual set.
 $n=m$ is a 00 set of the second category. So U A_n has the Baire property and is of the $n=m$ second category on every interval $\left[\frac{R}{n}, \frac{R+1}{n}\right]$ for $k = 0, \ldots, n - 1$ and $n \ge m$. 00 We have $\begin{array}{cccc} \mathsf{U} & \mathsf{A}_\mathbf{n} = (\mathsf{G} - \mathsf{P}_1) & \mathsf{U} & \mathsf{P}_2, \end{array}$ where G is an open set and $\begin{array}{cccc} \mathsf{P}_1, & \mathsf{P}_2 \end{array}$ are 00 of the first category. Then $[0,1] \setminus U$ $A_{\mathbf{n}} = [([0,1] \setminus G)$ \cup $P_{1}]$ \cap $[[0,1] \setminus P_{2}] \neq 1$. So $[0,1]\backslash G$ is a closed set of the second category. Hence, there exists an

interval $[a,b]$, $(a < b)$ such that $[a,b] \in [0,1] \backslash G$. There exist $n \ge m$ interval [a,b], (a < b) such that [a,b] \in [0,1]\G. There
and k \in N such that $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ \in [a,b]. Then $\bigcup_{n=m}^{\infty} A_n \cap \left[\frac{k}{n}, \frac{k+1}{n}\right]$ $\frac{k}{n}$, $\frac{k+1}{n}$ c [a,b]. Then $\bigcup_{n=m}^{\infty} A_n$ o $\big[\frac{k}{n}$, $\frac{k+1}{n}\big]$ G n $\left[\frac{k}{n}, \frac{k+1}{n}\right]$

In the general case (i.e. $E - [0,1] \neq 1$ and $E \cap [0,1] \in 1$) the proof is analogous.

Observe that the pair $(\ell \cap B, \pi \cap k)$ does not have property $(**)$. Let E \in [0,1] be a meager set of positive measure. Then E \in £ \cap B\n \cap K. But $\frac{1}{\infty}$ $\frac{1}{\infty}$ $\frac{1}{\infty}$ $E^{***} = \begin{bmatrix} 0 & 0 & 0 \\ m=1 & n=m \end{bmatrix}$ $\begin{matrix} \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \end{matrix}$ $\begin{matrix} \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \end{matrix}$ $\begin{matrix} \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \end{matrix}$

Now we shall consider the set $\{x \in [0,1]: \limsup_{n \to \infty} f(nx) \neq 1\}$ of $f(t)$.

$$
U(A) = \begin{matrix} \infty & \infty & \infty \\ n-1 & n-m & \end{matrix} \begin{bmatrix} \frac{1}{n} (A \cup (A+1) \cup ... \cup (A+n-1))] \\ = & \lim_{n} \sup \left[\frac{1}{n} (A \cup (A+1) \cup ... \cup (A+n-1)) \right].
$$

Let $A \subset [0,1)$. Put

It is easy to see that $U(A \cup B) = U(A) \cup U(B)$ for all $A, B \subset [0,1]$ and if $A \subseteq B$, then $U(A) \subseteq U(B)$. If $A_n \subseteq A$ for every n, then $\bm{\sigma}$ $U(A_n) \subset U(A)$. $\frac{0}{n=1}$

Let x be an irrational number and put $A = \{x\}$. We shall show that $U(A) = 0$. Suppose otherwise that there exists a real number z such that z ϵ U(A) = lim sup $\frac{1}{n}$ (A u (A + 1) u ... u (A + n - 1)). Then there exists a sequence ${n_m}$ tending to infinity such that

$$
z \in \bigcap_{m=1}^{\infty} \frac{1}{n_m} (A \cup (A + 1) \cup ... \cup (A + n_m - 1)).
$$

Hence there exist sequences ${n_m}$ and ${k_m}$, $k_m \nleq n_m - 1$ such that z = = $(x + k_m)/n_m$ for every m ϵ N. Let n_m , $\neq n_m$, We have

$$
(x + k_{m_1})/n_{m_1} = (x + k_{m_2})/n_{m_2}
$$

so $x = (k_{m_2} n_{m_1} - k_{m_1} n_{m_2})/(n_{m_2} - n_{m_1})$. But this is impossible because x is an irrational number. Consequently, $U(A) = \phi$. It is also easy to see that

if x is an irrational number, then all numbers of the form $\frac{x+k}{n}$, k $\le n$, are 00 00 00 different.

If $A = \bigcup_{n=1}^{\infty} A_n$, then $\bigcup_{n=1}^{\infty} U(A_n) \subset U(\bigcup_{n=1}^{\infty} A_n) = U(A)$ because of the $\begin{array}{ccc}\n\bullet & \bullet & \bullet & \bullet \\
\mathsf{U} & \mathsf{A}_n, & \text{then} & \mathsf{U}(\mathsf{A}_n) & \mathsf{C} & \mathsf{U}(\mathsf{U}) \\
\mathsf{n}=\mathsf{1} & & \mathsf{n}=\mathsf{1}\n\end{array}$ monotonicity of U. Observe that the reverse inclusion need not hold. We
 $\qquad \qquad \bullet$ shall find a sequence of sets {An} such that U(U An) * U U(An) . n=l n=l

Let z be an irrational number from the interval $[0,1]$. Put $x_n =$ 00 $= n \cdot z - [nz],$ $A_n = \{x_n\}$ for $n \in \mathbb{N}$ and $A = \bigcup_{n=1}^{\mathbb{U}} A_n$. Obviously, x_n is an irrational number; so $U(A_n) = \phi$ for $n \in \mathbb{N}$. We shall show that $z \in U(A)$. For every $n \in N$ we have

$$
z = \frac{x_n + [nz]}{n}
$$

z \in U(A). For every $n \in N$ we have
 $z = \frac{x_n + [nz]}{n}$

and $[nz] \le n$, because $z \le 1$. Consequently, $z \in \bigcap_{n=1}^{\infty} \frac{1}{n}$ (A u (A + 1) u ... u u ... u $(A + n - 1)$ c $U(A)$.

Observe that if $x = \frac{p_1}{q_1}$ (irreducible fraction), then $\frac{p}{q} \in U({x})$ if and only if q is a multiple of q_1 . Conversely, for all natural numbers example of the conversely, for all hatter pairs at the square of q_1 . Conversely, for all hatter r , $s \in N$ such that $r \times sq_1$ and $r \not| sq_1$ we have $\frac{r}{sq} \in U \left(\frac{p_1}{q_1}\right)$.

Theorem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a *&*-measurable, periodic function with period 1 and $M = 1$ -ess sup $f(t)$. Then Oátál

$$
0 \le t \le 1
$$

\n
$$
\{x \in [0,1) : \limsup_{n} f(nx) > M\} =
$$

\n
$$
= \bigcup_{k=1}^{\infty} U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1)).
$$

Proof. Put $A = \{x \in [0,1) : \lim_{n \to \infty} \sup f(nx) > M\}$ and let $x \in A$. Then $x \in [0,1)$ and $\lim_{n} \sup_{n} f(nx) > M$. There exists $k \in N$ such that lim sup f(nx) > M + $\frac{1}{k}$. So there exists a sequence $\{n_m\}$ tending to infinity $\begin{array}{cc} (0,1) & \text{and} & 1 \\ \text{sup } f(nx) > M + \frac{1}{k} \\ h & \text{that} & f(n,1) \end{array}$ such that $f(n_m x) > M + \frac{1}{k}$ for every $m \in N$. Observe that $x \in$ $\in U(f^{-1}((M+\frac{1}{k}, \infty)) \cap [0,1)).$ Indeed, let $f^{-1}((M+\frac{1}{k}, \infty)) \cap [0,1) = B.$ From the assumption that f is a periodic function, it follows that $f(n_m x) > M + \frac{1}{k}$ if and only if $n_m x \in B$ u $(B + 1)$ u ... u $(B + n_m - 1)$, i.e. $x \in \frac{1}{n_m}$ (Bu(B+l)u...u(B+n_m-l)). We have $f(n_m x) > M + \frac{1}{k}$ ssumption that f is a periodic function, it follows that
 $+\frac{1}{k}$ if and only if $n_m x \in B \cup (B + 1) \cup ... \cup (B + n_m - 1)$,
 $\frac{1}{n_m} (B \cup (B + 1) \cup ... \cup (B + n_m - 1))$. We have $f(n_m x) > M + \frac{1}{k}$
 $m \in N$. So $x \in \bigcap_{m=1}^{\infty} \frac{1}{n_m} (B \cup (B$ for every $m \in N$. So $x \in \bigcap_{m=1}^{\infty} \frac{1}{n_m} (B \cup (B + 1) \cup ... \cup (B + n_m - 1))$ c $\lim_{n \to \infty}$ (B u (B + 1) u ... u (B + n - 1) = U(B). This means that $A \subset \bigcup_{k=1}^{\infty} U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1)).$ $\begin{array}{lll} \cap & [0,1))\,. \ \circ & \ \text{or} & \ \text{or} & \ \text{or} & \text{or} \end{array}$

Conversely, let $x \in \bigcup_{k=1}^{\infty} U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$. There exists $k \in \mathbb{N}$ such that $x \in U(f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1))$. Put $f^{-1}((M + \frac{1}{k}, \infty)) \cap [0,1) = B$. We have $x \in U(B)$. So there exists a sequence $\{n_m\}$ tending to infinity such $\frac{1}{2}$ that $x \in \bigcap_{m=1}^{\infty} \frac{1}{n_m}$ (B u (B + 1) u ... u (B + n_m - 1)). This means that $n_m x \in$ ϵ B u (B + 1) u ... u (B + n_m - 1) for every m ϵ N. Hence $f(n_m x) > M + \frac{1}{k}$ (because f is a periodic function with period 1 and B + $\ell = f^{-1}((M + \frac{1}{k}, \infty))$ n n [ℓ , ℓ + 1) for every $\ell \in \mathbb{N}$. Consequently, lim_nsup f(nx) \geq lim_msup f(n_mx) $\geq M + \frac{1}{k} > M$ and $x \in A$.

The converse theorem also holds:

Theorem 3. If $B \in \mathcal{A}$ and if $B = \bigcup_{k=1}^{\infty} U(A_k)$ for some increasing sequence ${A_k}$ of \blacktriangle -measurable subsets of $[0,1)$, then there exists a \blacktriangle -measurable periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 1 such that

$$
\{x \in [0,1) : \limsup_n f(nx) > 4-\text{ess sup } f(t)\} = B.
$$

Proof. Put

$$
f(x) = \begin{cases} 1/min & \{e : x \in Ag\} & \text{if } x \in \bigcup_{e=1}^{\infty} Ag \\ 0 & \text{if } x \in [0,1) \setminus \bigcup_{e=1}^{\infty} Ag. \end{cases}
$$

We shall consider the periodic extension of f on R. Obviously $-$ ess sup f(t) = 0.
0 $\leq t \leq 1$

Observe that $\{x \in [0,1) : \limsup_{n} f(nx) > 0\} = B.$

Let $x \in B$. Evidently, $x \in [0,1)$. There exists $k \in N$ such that $x \in$ ϵ U(A_k) = lim sup $\frac{1}{n}$ (A_k u (A_k + 1) u ... u (A_k + n - 1)). So there exist k ϵ N and a sequence $\{n_m\}$ tending to infinity such that $n_m x \epsilon A_k$ u u (A_k + 1) u ... u (A_k + n_m - 1) for every m ϵ N. From the periodicity of f we have

$$
f(n_m x) = 1/\min \{\ell : n_m x \in Ag \cup (Ag + 1) \cup ... \cup (Ag + n_m - 1)\} \ge 1/k
$$

for every $m \in \mathbb{N}$. Hence lim sup $f(nx)$ in sup $f(n_{m}x) \ge 1/k$. Therefore sup $f(nx) \geq \lim_{m}$ Consequently, $B \subset \{x \in [0,1) :$ $x \in \{y \in [0,1) : \limsup f(ny) > 0\}.$ lim sup $f(nx) > 0$. n > 0 }. Consequently, B

sup f(ny) > 0}. Then lim

n

sequence {n_m} tending to infin

Let $x \in \{y \in [0,1) : \limsup f(ny) > 0\}.$ Then $\limsup f(nx) > 0.$ Hence there exist $k \in N$ and a sequence $\{n_m\}$ tending to infinity such that $f(n_m x) > 1/k$ for every m ϵ N. We have

$$
f(n_{m}x) = 1/min \{ 2 : n_{m}x - [n_{m}x] \in Ag \} > 1/k
$$

for every $m \in N$, and $\{A_{\ell}\}\$ is an increasing sequence of sets; so $n_{m}x \in$ ϵ A_k + [n_mx]. There are two cases:

for every
$$
m \in N
$$
, and $\{A_{\ell}\}$ is an increasing sequence of sets; so $n_m x \in$
\n $\in A_k + [n_m x]$. There are two cases:
\n1^{*} x = 0. Then the sequence $\{f(nx)\}$ is constant and equals $f(0)$.
\nThus, $\lim_{n} \sup f(nx) = f(0) > 1/k$. Hence $0 \in A_k$, $0 \in \frac{1}{2}(A_k \cup (A_k + 1)),...,$
\n $0 \in \frac{1}{n} (A_k \cup (A_k + 1) \cup ... \cup (A_k + n - 1)),...,$ Consequently,
\n ∞
\n $0 \in \prod_{n=1}^{\infty} (1/n) (A_k \cup (A_k + 1) \cup ... \cup (A_k + n - 1)) \in U(A_k) \in B.$

 2^* 0 $\lt x \lt 1$. Then $n_m x \lt n_m$ and $[n_m x] \le n_m - 1$ for every $m \in \mathbb{N}$. There exist $k \in N$ and a sequence $\{n_m\}$ tending to infinity such that $n_m x \in A_k$ u $(A_k + 1)$ u ... u $(A_k + n_m - 1)$ for $m \in N$. Hence $x \in \bigcap_{m=1}^{\infty} (1/n_m)(A_k \cup (A_k + 1) \cup ... \cup (A_k + n_m - 1)) \subset$ c $\lim_{n} \sup_{n} \frac{1}{n} (A_{k} \cup (A_{k} + 1) \cup ... \cup (A_{k} + n - 1)) = U(A_{k})$ c B.

Consequently $\{x \in [0,1) : \lim_{n \to \infty} \sup_{n} f(nx) > 0\} \subset B.$

Let $A \subset [0,1)$. But

$$
\mathcal{L}(A) = \mathop{0}\limits_{m=1}^{\infty} \mathop{0}\limits_{n \to m}^{\infty} \left[\frac{1}{n} (A \cup (A + 1) \cup ... \cup (A + n - 1)) \right] =
$$

= $\lim_{n} \inf \left[\frac{1}{n} (A \cup (A + 1) \cup ... \cup (A + n - 1)) \right].$

It is easy to see that $\hat{x}(A \cap B) = \hat{x}(A) \cap \hat{x}(B)$ for all A, $B \subset [0,1)$ and if $A \subseteq B$, then $\mathcal{L}(A) \subseteq \mathcal{L}(B)$. If $A = \bigcup_{n=1}^{\infty} A_n$, then $\bigcup_{n=1}^{\infty} \mathcal{L}(A_n) \subseteq \mathcal{L}(\bigcup_{n=1}^{\infty} A_n) =$ $=$ $\mathcal{L}(A)$ by the monotonicity of \mathcal{L} . The same example as for the operation U shows that the reverse inclusion need not hold.

Theorem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a *k*-measurable periodic function with period 1 and $M = 1 - e$ ss sup $f(t)$. Then $0 \leq t \leq 1$

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\n{
$$
x \in [0,1)
$$
: $\limsup_{n} f(nx) \langle M \rangle = \bigcup_{k=1}^{\infty} \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$

Proof. Put $A = \{x \in [0,1) : \limsup_{n \to \infty} f(nx) \leq M\}$ and let $x \in A$. Then n $x \in [0,1)$ and $\lim_{n} \sup f(nx) < M$. There exists $k \in \mathbb{N}$ such that **Theorem 4.** Let $f: \mathbb{R} \to \mathbb{R}$ be a *d*-measurable periodic function with
period 1 and $M = 3$ -ess sup $f(t)$. Then
 $0 \le t \le 1$
 $\{x \in [0,1) : \limsup_{n \to \infty} f(nx) \le M\} = \frac{1}{k-1} \pounds(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$
Proof. Put $A = \{x \in [$ $x \in [0,1)$ and $\lim_{n \to \infty} \sup f(nx) \le M$. There exists $k \in \mathbb{N}$ such that
 $\lim_{n \to \infty} \sup f(nx) \le M - \frac{1}{k}$. So there exists $n_0 \in \mathbb{N}$ such that $f(nx) \le M - \frac{1}{k}$

for $n \ge n_0$. Observe that $x \in \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. period 1 and $M = 3-\text{ess} \sup f(t)$. Then
 $0.4 \text{ if } t \in [0,1) : \limsup_{n \to \infty} f(nx) \leq M$ = $\frac{1}{k-1} \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$.

Proof. Put $A = \{x \in [0,1) : \limsup_{n \to \infty} f(nx) \leq M\}$ and let $x \in A$. Then
 $x \in [0,1)$ and $\limsup_{n \to \infty} f(nx) \leq M$ **Proof.** Put $A = \{x \in [0,1) : \limsup_{n \to \infty} f(nx) \times M\}$ and let $x \in A$. Then
 $x \in [0,1)$ and $\limsup_{n \to \infty} f(nx) \times M$. There exists $k \in N$ such that
 $\limsup_{n \to \infty} f(nx) \times M - \frac{1}{k}$. So there exists $n_0 \in N$ such that $f(nx) \times M - \frac{1}{k}$

fo { $x \in [0,1)$: $\limsup_{n} f(nx) \langle M \rangle = \prod_{k=1}^{n} \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$

Proof. Put $A = \{x \in [0,1) : \limsup_{n \to \infty} f(nx) \langle M \rangle \}$ and let $x \in A$. Then
 $x \in [0,1)$ and $\limsup_{n} f(nx) \langle M \rangle$. There exists $k \in N$ such that
 $\limsup_{n} f(nx) \langle M - \frac{$ { $x \in [0,1)$: $\limsup_{n \to \infty} f(nx) \le M$ } = $\int_{k=1}^{0} \ell(f^{-1}((-\infty, M - \frac{1}{K})) \cap [0,1)).$

Proof. Put $A = \{x \in [0,1) : \limsup_{n \to \infty} f(nx) \le M\}$ and let $x \in A$. Then
 $x \in [0,1)$ and $\limsup_{n \to \infty} f(nx) \le M$. There exists $k \in N$ such that
 $\limsup_{n \to$ have $f(nx) \leq M - \frac{1}{k}$ for $n \geq n_0$. Hence $x \in \lim_{n \to \infty} \inf \frac{1}{n} (B \cup (B + 1) \cup ... \cup$ σ (B + n - 1)) = $\ell(B)$. This means that $A \subset \bigcup_{k=1}^{\infty} \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$

+ n - l)) = $\mathcal{L}(B)$. This means that $A \subset U$ $\mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$
Conversely, let $x \in U$ $\mathcal{L}(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1)).$ There exists $k \in N$ Conversely, let $x \in \bigcup_{k=1}^{\infty} \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. There exists $k \in \mathbb{N}$
such that $x \in \ell(f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1))$. Put $f^{-1}((-\infty, M - \frac{1}{k})) \cap [0,1) = B$.
We have $y \in \ell^{(p)}$, Thus there exists $x \in \mathbb{N}$ suc We have $x \in \mathcal{X}(B)$. Thus there exists $n_0 \in \mathbb{N}$ such that $x \in \frac{1}{n}$ (B \cup (B + 1) \cup u ... u $(B + n - 1)$ for $n \ge n_0$. This means that $nx \in (B \cup (B + 1) \cup ... \cup$ $u (B + n - 1)$) for $n \ge n_0$. Therefore $f(nx) < M - \frac{1}{k}$ (because f is a

periodic function with period 1 and $B + 8 = f^{-1}((-\infty, M - \frac{1}{k}))$ n $[8, 8 + 1)$ for every $\ell \in \mathbb{N}$. Consequently, lim sup $f(nx) \leq M - \frac{1}{k} \leq M$ and $x \in A$. d $B + 8 = f'$ ((- ∞)

n K (nx) $\leq M - \frac{1}{k}$

The converse theorem also holds.

Theorem 5. If $B \in \mathcal{A}$ and if $B = \bigcup_{k=1}^{\infty} \mathcal{L}(A_k)$ for some increasing sequence ${A_k}$ of \triangleleft -measurable subsets of $[0,1)$, then there exists a \triangleleft -measurable periodic function $f : \mathbb{R} \to \mathbb{R}$ with period 1 such that

$$
\{x \in [0,1) : \limsup_n f(nx) \leq 4-\text{ess sup } f(t)\} = B.
$$

Proof. Put

$$
f(x) = \begin{cases} (-1)/\min \{f : x \in Ag\} & \text{if } x \in \bigcup_{\ell=1}^{\infty} Ag \\ 0 & \text{if } x \in [0,1) - \bigcup_{\ell=1}^{\infty} Ag. \end{cases}
$$

We shall consider the periodic extension of f on R. Obviously, **4-ess sup f(t) = 0.
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Observe that $\{x \in [0,1) : \limsup_n f(nx) < 0\} = B.$

Let $x \in B$. Evidently, $x \in [0,1)$. There exists $k \in N$ such that $x \in$ $\epsilon \mathcal{L}(A_k) = \liminf_{n \to \infty} \frac{1}{n} (A_k u (A_k + 1) u ... u (A_k + n - 1)).$ So there exist $k \in N$ and $n_0 \in N$ such that $nx \in (A_k \cup (A_k + 1) \cup ... \cup (A_k + n - 1))$ for $n \geq n_0$. From the periodicity of f we have

$$
f(nx) = (-1)/min
$$
 { $\ell : nx \in A_{\ell} \cup (A_{\ell} + 1) \cup ... \cup$
 $\cup (A_{\ell} + n - 1)) \le (-1)/k$

for $n \ge n_0$. Hence $\limsup_{n} f(nx) \leq (-1)/k < 0$. Therefore $x \in \{y \in [0,1) :$: lim sup $f(ny) < 0$. Consequently, $B \subset \{x \in [0,1) : \limsup_{n} f(nx) < 0\}.$ $f(nx) = (-1)/min$ { $f : nx \in Ag$ u (A $f + 1$) u ... u

u (A $f + n - 1$)) $f = (-1)/k$

m $\Rightarrow n_0$. Hence $\lim_{n \to \infty} \sup_{n \to \infty} f(nx) = (-1)/k < 0$. Therefor

sup $f(ny) < 0$. Consequently, $B \subset \{x \in [0,1) : \lim_{n \to \infty} f(x) \leq 0\}$. : have
 $(0.4e + 1)$ 0 ... 0
 $(1)/k$
 $(-1)/k < 0$. Therefore $x \in \{y \in [0,1) :$
 $(x \in [0,1) : \limsup_{n} f(nx) < 0\}.$

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Let $x \in \{y \in [0,1) : \limsup_{n \to \infty} f(ny) < 0\}.$ Then $\limsup_{n \to \infty} f(nx) < 0.$ sup $f(ny) < 0$. Then $\lim_{n \to \infty}$
d $n_0 \in N$ such that $f(nx) <$ Hence there exist $k \in N$ and $n_0 \in N$ such that $f(nx) < (-1)/k$ for $n \ge n_0$. We have $f(nx) = (-1)/min$ { $\ell : nx - [nx] \in Ag$ } < $(-1)/k$ for $n \ge n_0$ and ${A_{\ell}}$ is an increasing sequence of sets. Hence $nx \in A_k + [nx]$ for $n \geq n_0$. There are two cases:

1^{*} x = 0. Then the sequence
$$
\{f(nx)\}
$$
 is constant and equals $f(0)$. Thus
\n
$$
\limsup_{n} f(nx) = f(0) \langle (-1)/k. \text{ Hence } 0 \in A_k,
$$
\n0 $\in \frac{1}{2} (A_k \cup (A_k + 1)),..., 0 \in \frac{1}{n} (A_k \cup (A_k + 1) \cup ... \cup (A_k + n - 1)),...$
\nConsequently, $0 \in \bigcap_{n=1}^{\infty} (1/n) (A_k \cup (A + 1) \cup ... \cup (A_k + n - 1)) \in \mathcal{L}(A_k) \subset B.$

2° 0 < x < 1. Then
$$
nx < n
$$
 and $[nx] \le n - 1$ for every $n \in \mathbb{N}$. There exist $k \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $nx \in (A_k \cup (A_k + 1) \cup \ldots \cup (A_k + n - 1))$ for $n \ge n_0$. Hence $x \in \bigcap_{n=0}^{\infty} (1/n)(A_k \cup (A_k + 1)) \cup \ldots \cup (A_k + n - 1)) =$ $= \mathfrak{L}(A_k) \subset B.$

Consequently, $\{x \in [0,1) : \limsup_{n} f(nx) < 0\} \subset B$.

References

[1] Scottish book, Solution by S. Banach of Problem 162 of H. Steinhaus.

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