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MAXIMAL ADDITIVE FAMILIES FOR SOME CLASSES OF DARBOUX FUNCTIONS

Preliminaries. Let $C^+(f,x)$ and $C^-(f,x)$ denote the set of all right-side and left-side limit numbers of the function f at the point x. For any subset M of the plane $\mathbb{R}^2$, $cl(M)$ denotes the closure of M and $card(M)$ denotes the cardinality of M. No distinction is made between a function and its graph.

In [2] Bruckner and Ceder described what it means for a real function to be Darboux at a point. We say that a function f is Darboux from the right-side [left-side] at a point x (we write $x \in D_+(f)$, $x \in D_-(f)$ respectively) if and only if

1° $f(x) \in C^+(f,x)$ $[f(x) \in C^-(f,x)]$; and

2° whenever $a,b \in C^+(f,x)$ $[a,b \in C^-(f,x)]$ and y is any point between a and b, then for every $\varepsilon > 0$ exists a point $\xi \in (x,x+\varepsilon)$ $[\xi \in (x-\varepsilon,x)]$ such that $f(\xi) = y$.

In [3] Császár showed that a function is Darboux if and only if it is Darboux at each point.

By D^c we denote the class of Darboux functions whose upper and lower boundary functions are continuous. By D^* we denote the class of functions which take on every real value in every interval and by D^{**} we denote the class of functions which take on every real value c-times in every interval, where c denotes the cardinality of the continuum.

It is clear that $D^{**} \subset D^* \subset D^c$.

For a given family F of real functions let $M(F)$ denote the class of all functions g such that $f \in F$ implies $f + g \in F$. This class is called the maximal additive family for F. It is known [1] that the family of continuous functions C is the maximal additive family for the class of Darboux Baire 1 functions DB_1 and the family of constant functions K is the maximal additive family for the class of Darboux functions D.

Definition 1. We say that $g \in \mathfrak{a}$ if g is a continous function and there exists a sequence of open intervals $\{I_k\}$ such that

1° $\bigcup_{k=1}^{\infty} I_k$ is dense in $\mathbb{R}$; and

2° $g|_{I_k}$ is constant for every k.

Theorem 1. $M(D^c) = \mathfrak{a}$.

Proof. Let $f \in D^c$, $g \in \mathfrak{a}$ and $x_0 \in \mathbb{R}$. Without loss of generality we may assume that $g(x_0) = 0$. We shall prove that $x_0 \in D_+(f+g)$. If $C^+(f,x_0)$ is a one-point set, then x_0 is a point of right-side continuity of f. Hence $x_0 \in D_+(f+g)$. Now we consider the case in which $C^+(f,x_0)$ is nondegenerate. Let $a,b \in C^+(f,x_0) = C^+(f+g,x_0)$, $\varepsilon > 0$ and $z \in (a,b)$. There exist a_1, b_1, ε_1 such that

1° $a < a_1 < z < b_1 < b$;

2° $0 < \varepsilon_1 \leq \varepsilon$; and

3° $\{(x,y) : x_0 < x < x_0 + \varepsilon_1,\ a_1 < y < b_1\} \subset cl(f)$.

Hence $(a_1,b_1) \subset C^+(f,x)$ for every $x \in (x_0,x_0 + \varepsilon_1)$. There exists an open interval I_k and a point z_0 such that

1° $g(x) = z_0$ for every $x \in I_k$;

2° $(x_0,x_0 + \varepsilon_1) \cap I_k \neq \emptyset$; and

3° $|z_0| < \frac{1}{2}\min\{|z-a_1|, |z-b_1|\}$.

Let $(r,s) = (x_0,x_0 + \varepsilon_1) \cap I_k$. Since $z - z_0 \in (a_1,b_1) \subset C^+(f,r)$, $((r,s) \times \{z-z_0\}) \cap f \neq \emptyset$ and $((r,s) \times \{z\}) \cap (f+g) \neq \emptyset$. In a similar way we can prove that $x_0 \in D_-(f+g)$.

It is clear that the upper and lower boundary functions of $f+g$ are continuous. We have shown that $\mathfrak{a} \subset M(D^c)$.

We will show that $M(D^c) \subset \mathcal{A}$. Let $g \notin \mathcal{A}$. It is easy to show that if g is not continuous, then there exists $f \in D^C$ such that $f+g \notin D^C$. Indeed, if $g \notin D^C$, then we put $f(x) = 0$; if $g \in D^C$, then there exists a point x_0 such that the set $C(g,x_0)$ is nondegenerate. Now we put $f(x) = -g(x)$ for $x \neq x_0$ and $f(x_0) = -y$, where $y \in C(g,x_0)\setminus\{g(x_0)\}$. We need only consider the case: g is continuous and there exists an interval I such that g is nonconstant on every subinterval of the interval I. Let $M = \sup\{g(x):x \in I\}$ and $m = \inf\{g(x):x \in I\}$. The set $g^{-1}(y) \cap I$ is nowhere dense for every $y \in [m,M]$. Let $\{A_y\}_{y\in[m,M]}$ denote a family of sets which are pairwise disjoint, dense in $\mathbb{R}$ and such that $I \cap A_y \cap g^{-1}(y) = \emptyset$. We define the function f as follows:

$$f(x) = \begin{cases} y & \text{if } x \in A_y \\ m & \text{otherwise.} \end{cases}$$

It is easy to see that $f \in D^C$ but $f+g$ is not a Darboux function.

Definition 2. We say that a function $g \in \mathcal{B}$ iff there exists a sequence of open intervals $\{I_k\}$ such that

1° $\bigcup_{k=1}^{\infty} I_k$ is dense in $\mathbb{R}$

2° $g|_{I_k}$ is constant for every k.

Theorem 2. $M(D^*) = \mathcal{B}$.

Proof. Let I be an open interval and let $f \in D^*$, $g \in \mathcal{B}$ and $z \in \mathbb{R}$. There exists an interval $I_0 \subset I$ such that $g|_{I_0}$ is constant. Let $g(x) = z_0$ for every $x \in I_0$. There exists $x_0 \in I_0$ such that $f(x_0) = z - z_0$. Hence $(f+g)(x_0) = z$. We have shown that $f+g \in D^*$.

If g is nonconstant on some interval I, then by Theorem 1 [2], there exists a function $d \in D^*$ such that $g+d \notin D^*$. This completes the proof.

Definition 3. We say that $g \in \mathcal{C}$ if there exists a sequence of open intervals $\{I_k\}$ and a sequence of sets $\{A_k\}$ such that

1° $\bigcup_{k=1}^{\infty} I_k$ is dense in $\mathbb{R}$;

2° $A_k \subset I_k$ and $\mathrm{card}(A_k) < c$; and

3° $g|_{I_k - A_k}$ is constant for every k.

Theorem 3. $M(D^{**}) = \mathcal{C}$.

Proof. Let I be an open interval and let $f \in D^{**}$, $g \in \mathcal{C}$ and $z \in \mathbb{R}$. There exists an interval $I_0 \subset I$ and a set $A \subset I_0$ such that

1° $g|_{I_0 \setminus A}$ is constant; and

2° $\mathrm{card}(A) < c$.

Let $g(x) = z_0$ for every $x \in I_0 \setminus A$. There exists a set $B \subset I_0$ such that

1° $f(x) = z - z_0$ for every $x \in B$; and

2° $\mathrm{card}(B) = c$.

Hence $(f+g)(x) = z$ for every $x \in B \setminus A$. We have shown that $f+g \in D^{**}$.

If $g \notin \mathcal{C}$, then there exists an open interval I_0 such that for every subinterval $I \subset I_0$ and every real λ we have $\mathrm{card}(\{x \in I: g(x) \neq -\lambda\}) = c$. By Theorem 3, [2], there exists a function $f \in D^{**}$ such that $(f+g)(x) \neq 0$ for every $x \in I_0$ This completes the proof.

References

[1] A.M. Bruckner, Differentiation of real functions, Lect. Notes in Math. 659, Springer-Verlag, 1978.

[2] A.M. Bruckner, J.G. Ceder, On the sum of Darboux functions, Proc. Amer. Math. Soc. 51 (1975), 97-102.

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