TOPICAL SURVEY

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POROSITY AND σ-POROSITY

1. Introduction. The notion of the porosity of a set $E \subset R$ at a point $x \in R$ concerns the size of "holes" in the set E near to x. A porous set P is not only nowhere dense but it is small in a stronger sense: near to each point $x \in P$ are "holes" in P which are in a sense big.

Porosity computations arise naturally in some problems in Real Analysis, especially in the differentiation theory, and were used by many authors (cf. 3.B). Porosity was used under a different nomenclature by Denjoy [16], [17] in the study of the second order symmetric derivative (cf. [15]), where an exposition of Denjoy's results can be found. Underline the following fact: Denjoy's bilateral index which is his main notion is not equivalent to the ordinary bilateral porosity, but to the symmetrical porosity (cf. 2.B) which is not used in recent works. Porosity was also applied by Khintchine [41] (for this application it is necessary to consult [10], where some corrections are made).

The investigation of σ-porous sets was started in 1967 by Dolženko [18], who also at first used the term "porous set" ("poristoe množestvo" in Russian). He proved that certain exceptional sets in the theory of the cluster sets are σ-porous (= are countable unions of porous sets). Dolženko observed that the class of σ-porous sets is a subclass of the class of measure-zero, first category sets and stated that it is a proper subclass. This basic fact justifies the theorems which assert that some exceptional sets are not only measure-zero first category sets but are also σ-porous. There is also another argument for such theorems: it is frequently easier to prove σ-porosity than to give two proofs, one on category and the second on the measure.

After the Dolženko's work a number of similar theorems appeared in the cluster sets theory (cf. 3.A). In 1978 Belna, Evans and Humke [5] proved the first theorem which asserts that a natural exceptional set in the differentiation theory is σ-porous. For further results of this sort see 3.B and 3.D.

Proofs of the existence of a measure-zero first category non σ-porous set are relatively difficult. The first published proof is contained in [82]; now there exist several different proofs of this basic fact (cf. 2.A, 4.A, 4.C, 4.F and 5).

The existence of a number of theorems using σ-porosity shows that the σ-ideal of σ-porous sets is of some importance. Thus there exists a motivation for an investigation of structural properties of this σ-ideal. It was started in [82], where the notion of porosity and σ-porosity was considered in general metric spaces. In the same article also the Yanagihara's [74] notion of "porosity (q)" was investigated and further generalized. At the present time there exists a number of further notions which are variations of the notion of porosity.

Many theorems using porosity and σ-porosity in the first order differentiation theory on the real line are presented (frequently with simplified proofs) in the Thomson's monograph [66]. This book contains also a material (30 pp.) on porosity and is frequently used and quoted below.

2. Definitions and basic comments.

2.A Ordinary and strong porosity in R.

Definition 2.1. Let $E \subset R$ be a set and let I be an interval. Then we denote by $\lambda(E,I)$ the length of the largest open subinterval of I which does not intersect E.

Definition 2.2. Let $E \subset R$, $x \in R$. Then we define:

(i) the porosity of E at x as

$$p(E,x) = \limsup_{h \to 0^+} \frac{\lambda(E,(x-h,x+h))}{h},$$

(ii) the right porosity of E at x as

$$p^+(E,x) = \limsup_{h \to 0^+} \frac{\lambda(E,(x,x+h))}{h} \quad \text{and}$$

(iii) the left porosity of E at x as

$$p^-(E,x) = \limsup_{h \to 0^+} \frac{\lambda(E,(x-h,x))}{h}.$$

Remark 2.3.

(a) Some authors use a definition of the porosity which differs from the above in the trivial case $x \notin \bar{E}$, namely they put $p(E,x) = \max(p^+(E,x),p^-(E,x))$.

(b) Clearly $0 \leq p^+(E,x) \leq 1$, $0 \leq p^-(E,x) \leq 1$, and in the case $x \in \bar{E}$ also $0 \leq p(E,x) \leq 1$ (if $x \notin \bar{E}$, then $p(E,x) = 2$). If $x \in \bar{E}$, then $p(E,x) = \max(p^+(E,x),p^-(E,x))$.

(c) Clearly $p(E,x) = p(\bar{E},x)$, $p^+(E,x) = p^+(\bar{E},x)$.

Definition 2.4. Let $E \subset R$, $x \in R$. Then we say that

(i) E is porous at x if $p(E,x) > 0$,

(ii) E is porous on the right at x if $p^+(E,x) > 0$,

(iii) E is strongly porous at x if $p(E,x) \geq 1$,

(iv) E is strongly porous on the right at x if $p^+(E,x) = 1$,

(v) E is bilaterally porous at x if it is porous both on the right and on the left at x.

The notions of the left porosity, the left strong porosity, and bilateral strong porosity are defined in the obvious way.

Remark 2.5. (a) If E is strongly porous at x, then it is strongly porous on the right or on the left at x. If in addition $x \in \bar{E}$, then $p(E,x) = 1$.

(b) If E is porous at x, then $x \notin \operatorname{Int} \bar{E}$ and x is not the point of outer density for E.

(c) The porosity notions are local notions.

Definition 2.6. A set $E \subset R$ is said to be

(i) porous, if it is porous at each of its points and

(ii) σ-porous, if it is a countable union of porous sets.

The notions of a strongly porous set, a bilaterally porous set, a bilaterally strongly porous set, a σ-strongly porous set, a σ-bilaterally porous set, a σ-bilaterally strongly porous set are defined in the obvious way.

The following simple proposition is due to Denjoy [17] (cf. [66], p. 188).

Proposition 2.7. If $E \subset R$ is closed and nowhere dense, then the set of points in E at which E is bilaterally strongly porous is residual in E.

The following important fact is easy to prove.

Theorem 2.8. For subsets of R the following assertions hold.

(i) A porous set is nowhere dense and has measure zero.

(ii) The class of σ-porous sets is a σ-ideal contained in the σ-ideal of measure-zero, first category sets.

Remark 2.9. A porous set $E \subset R$ has measure zero by Remark 2.5(b), since almost all points of an arbitrary set E are points of outer density for E (see [59], p. 129). It is not correct to use directly the Lebesgue density theorem for measurable sets, since apriori we do not know the measurability of E. Another (more direct) proof is via Proposition 4.24 below.

The following theorem is a fact of basic importance.

Theorem 2.10. There exists a perfect measure-zero set $E \subset R$ which is not σ-porous.

The following remarks concern proofs of this basic fact.

Consider at first the best known perfect measure-zero set - the classical Cantor ternary set C. It is easy to see that $p(C,x) \geq 1/2$ for each $x \in C$ and therefore C is porous. Now it is natural to consider more general symmetric perfect sets.

Definition 2.11. Let $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ be a sequence with $0 < \alpha_n < 1$. Then the symmetric perfect set $C(\alpha) \subset [0,1]$ can be defined similarly as the Cantor ternary set. The only difference is that in the n-th step of the construction we delete from the 2^{n-1} remaining closed intervals of length d_n the concentric open intervals of length $\alpha_n d_n$. Thus $C = C(1/3, 1/3, \ldots)$.

Remark 2.12. For more formal definition see [36] (cf. also [3], [66], and [39], where a slightly different notation is used).

If $\alpha_n \rightarrow 0$, then it is not too difficult to prove that the set of points $x \in C(\alpha)$, at which $C(\alpha)$ is not porous, is dense in $C(\alpha)$. In addition, an easy computation yields that $\mu(C(\alpha)) = 0$ iff $\sum_{n=1}^{\infty} \alpha_n = \infty$. Therefore $C(\alpha)$ for $\alpha_n = (n+1)^{-1}$ is the most natural candidate for an example of a perfect measure-zero non σ-porous set. The following theorem implies that in fact this set is not σ-porous.

Theorem 2.13 ([37], [66]). The symmetric perfect set $C(\alpha)$ is non σ-porous iff $\alpha_n \rightarrow 0$.

Nevertheless, the complete proof of Theorem 2.13 given in [34] is rather complicated. The proofs in [37] and [66] are sketched only and seem to be incomplete. Note that Humke [34] considers perfect sets which are generated by a more general process than symmetric perfect sets and thus he proves a theorem which is more general than Theorem 2.13.

Further note that the single information that the set $N(F)$ of points, at which a given perfect set $F \subset R$ is not porous, is dense in F does not imply non σ-porosity of F. Moreover, it seems that no reasonable "topological" information concerning $N(F)$ and F implies non σ-porosity of F.

In [82] a construction of a perfect set, more complicated than the construction of a symmetric perfect set but similar to it, is chosen, which makes the proof of non σ-porosity easier in comparison with the proof in [34].

Tkadlec [67] generalized the method of [82] and solved in this way several open problems. Although the method of [67] and [82] is not too complicated, it is necessary to prefer new methods of constructions of non σ-porous sets given by Konjagin (cf. Theorem 5.1 below) and by Foran [26] (cf. 4.A and 4.D), which are more analytical.

The existence of a porous perfect set which is not σ-strongly porous is obtained in [82] as a consequence of a more general proposition. In fact, the Cantor ternary set has this property, as the following theorem shows.

<u>**Theorem 2.14**</u> ([66]). A symmetric perfect set $C(\alpha)$ is non σ-strongly porous iff $\limsup \alpha_n < 1$.

The following structural proposition for σ-porous sets ([82], cf. [66]) is a very useful technical tool.

<u>**Proposition 2.15**</u>. Let $c < 1$. Then any σ-porous set $A \subset R$ may be expressed as the union of a sequence of sets $\{A_n\}$ such that the porosity of each A_n at each of its points is at least c.

<u>2.B Denjoy's index and symmetric porosity</u>.

Denjoy used the following definitions of indexes in the case when E is a perfect set (cf. [15], pp. 87-88).

Definition 2.16. Let $E \subset R$ and $x \in R$. Then the right index of E at x, $i^+(E,x)$, is defined to be the infimum of all numbers r for which a sequence of points $\{x + h_n\}$ in E can be found with $h_n > 0$, $h_n \to 0$ and $1 < h_n/h_{n+1} < r$. The left index is defined in the obvious way.

Definition 2.17. Let $E \subset R$, $x \in R$. Then the index of E at x, $i(E,x)$, is defined to be the infimum of all numbers r for which a sequence of points $\{x + h_n\}$ in E can be found with $h_n \to 0$ and $1 < |h_n/h_{n+1}| < r$.

To the index, the following notion of symmetric porosity corresponds [63].

Definition 2.18. Let $E \subset R$, $x \in R$ and $r > 0$. Then we define:

(i) $s(E,x,r)$ as the supremum of all numbers $h > 0$ for which there exists $t > 0$ such that $t + h \leq r$, $(x+t,x+t+h) \cap E = \emptyset$ and $(x-t-h,x-t) \cap E = \emptyset$;

(ii) the symmetric porosity of E at x as

$$p^s(E,x) = \limsup_{r \to 0^+} \frac{s(E,x,r)}{r} .$$

Remark 2.19. It is easy to see that $p^s(E,x) = p^+(E^s,x)$, where $E^s = E \cup (2x - E)$ is the symmetrization of E about the point x.

The notions of symmetrically porous sets, strongly symmetrically porous sets etc. are defined in the obvious way (cf. Definitions 2.4, 2.6).

Of course, symmetric porosity is more restrictive than the bilateral porosity. It seems to be probable that there exist strongly bilaterally porous sets which are not σ-symmetrically porous and that the analogue of Proposition 2.15 for symmetric porosity does not hold.

The connection between indexes and porosities is given in the following lemma ([15] and [63], pp. 415-416).

Lemma 2.20. If $E \subset R$ and $x \in R$, then

$$i^+(E,x) = (1 - p^+(E,x))^{-1} \quad \text{and} \quad i(E,x) = (1 - p^s(E,x))^{-1} ,$$

where $0^{-1} = \infty$.

Finally note that Denjoy proved some general properties of the index; his result can be reformulated in the porosity language as follows. For the original formulation see Corollary 2 of [15], p. 96.

Proposition 2.21. Let a perfect set $P \subset R$ be strongly porous at each of its points. Then there exists a point $x \in P$ at which P is strongly symmetrically porous.

2.C Porosity in general metric spaces.

In this section we suppose that P is a metric space. The notion of set porosity was generalized to P in [82]. A slightly different definition was used in [2] (cf. Definition 2.31 below).

The open ball with the center $x \in P$ and the radius $r > 0$ will be denoted by $B(x,r)$.

Definition 2.22. Let $M \subset P$, $x \in P$ and $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $z \in P$ such that $B(z,r) \subset B(x,R) - M$ by $\gamma(x,R,M)$. The number $p(M,x) = 2 \limsup_{R \to 0^+} \frac{\gamma(x,R,M)}{R}$ is called the porosity of M at x.

Remark 2.23. In the definition of porosity, the coefficient 2 is usually omitted (cf. [82], [56], [26]). The present definition is a direct generalization of the usual one-dimensional porosity. Dolženko [18] considered porosity on surfaces in Euclidean spaces.

Porous, σ-porous, strongly porous and σ-strongly porous sets in P are defined in the same manner as in 2.A.

The notion of strong porosity in more dimensions is not so natural as in one dimension. Nevertheless, it is easy to see that the following generalization of Denjoy's Proposition 2.7 holds (for the proof in R^n cf. [26], p. 195).

Proposition 2.24. Let E be a nowhere dense set in P. Then the set of all points of E at which E is strongly porous is residual in E.

Also the structural Proposition 2.15 holds for $A \subset P$ (cf. Proposition 4.4 which provides a further generalization).

If $P = R^n$ or, more generally, P is a finite dimensional Banach space, then Theorem 2.8 still holds; in particular, each σ-porous set is a first category null set. In an arbitrary P we easily obtain that a σ-porous set is of the first category, but we have no "measure smallness". For example, the next proposition follows from results of [56].

Proposition 2.25. Let H be a separable infinite-dimensional Banach space and let ν be a Radon measure on H. Then there exists a convex compact σ-porous set K with $\nu(K) > 0$.

Remark 2.26. In fact, it is proved in [56] that K is porous. Moreover, it is porous in a more restrictive sense (it is e.g. totally porous - see Theorem 3.13 below).

The following lemma is a special case of [82], Lemma 3.4, which deals with generalized porosities.

Lemma 2.27. Let $M \subset R$ and put $N = M \times R^n \subset R^{n+1}$. Then N is a porous subset (or a σ-porous subset, or a strongly porous subset, or a σ-strongly porous subset) of R^{n+1} iff M is of the same type as a subset of R.

Using this lemma and Theorem 2.10 we obtain the following

Theorem 2.28. In an arbitrary Euclidean space there exists a non-σ-porous perfect set of measure zero.

The following lemma can be proved by the same method as Lemma 2.27.

Lemma 2.29 ([56]). Let X be a Banach space, $0 \neq p \in X^*$ and let $K \subset R$ be a nowhere dense non σ-porous set. Then $p^{-1}(K)$ is a nowhere dense non σ-porous subset of X.

Consequently we obtain the following important fact.

Proposition 2.30. The notion of a σ-porous set is strictly more restrictive than the notion of a first category set in each Banach space P.

Agronsky and Bruckner [2] proved the same proposition (using a slightly different notion of porosity) for more general metric spaces P — closed non-locally compact convex subsets of a separable Banach space. However, it seems that it is not known at which more general metric spaces Proposition 2.30 holds. In particular, it is probably not proved in the space $\mathcal{C}$ of all convex bodies in R^n. Note that σ-porosity in $\mathcal{C}$ was used in [94]; cf. 3.C.

In [2], the following definitions are used.

Definition 2.31. Let $E \subset P$, $x \in E$ and let $S \subset P$ be an open ball such that x is in the boundary of S. Then E is said to be porous at x with respect to S if there exists a $c > 0$ such that for every $\varepsilon > 0$ there exist open balls $S_1 \subset S_2 \subset S$ such that x is a boundary point of S_2, $S_1 \cap E = \emptyset$ and $\varepsilon >$ diameter $S_1 \geq c$ (diameter S_2).

If each $x \in E$ is porous w.r.t. some sphere, we say that E is a porous set. If E is porous at $x \in E$ w.r.t. every open ball containing x in its boundary, we say that E is totally porous at x.

A related very restrictive notion of the directional porosity in Banach spaces is used in [88].

Definition 2.32. Let X be a Banach space. We say that $E \subset X$ is directionally porous at $x \in X$ if there exist $0 \neq v \in X$, $p > 0$, $t_n \searrow 0$ and $r_n \searrow 0$ such that

$$B(a + t_n v, pr_n) \subset B(a, r_n) - E \ .$$

2.D Generalized porosity defined by a function.

There exist several slightly different notions of a generalized porosity defined by an increasing function $g : [0,\infty) \longrightarrow [0,\infty)$ or by a family of such functions ([82], [12], [31], [66], [92]). Probably the most interesting case (different from ordinary porosity) when $g(x) = x^q$, $0 < q < 1$, was firstly considered and applied by Yanagihara [74] (he used the term "set of σ-porosity (q)").

In [82], the (g)-porosity was considered in a general metric space P. In the following the symbols P, B(x,r) and $\gamma(x,R,M)$ have the same meaning as in 2.C.

Let G be the system of all real functions g which are increasing and continuous on $[0,h)$ for some $h > 0$, with $g(0) = 0$. Functions from G will be called "porosity functions". Note that some authors used the following definitions also for more general (e.g. discontinuous) functions.

Definition 2.33. Let $g \in G$, $M \subset P$ and $x \in P$. We say that M is (g)-porous at x if $\limsup_{R \to 0^+} \frac{1}{R}\, g(\gamma(x,R,M)) > 0$.

Definition 2.34. Let $g \in G$, $H \subset G$, $M \subset P$ and $x \in P$. Then we say that M is $\langle g \rangle$-porous at x if there exists a sequence of spheres $\{B(s_n,r_n)\}$ such that $s_n \longrightarrow x$, $B(s_n,r_n) \cap M = \emptyset$ and $g(r_n) > \rho(x,s_n)$. We say that M is $\langle H \rangle$-porous at x if it is $\langle h \rangle$-porous at x for each $h \in H$.

The above definitions are used in [82]. Similar definitions on R are given in [12], [31], [66], [92]. The symbol $\lambda(E,I)$ has the same meaning as in 2.A.

Definition 2.35 ([12]). Let $E \subset R$, $x \in R$ and let ψ be a porosity function. We say that E is (ψ)-porous on the right at x if $p_{\psi}^{+}(E,x) = \limsup_{h \to 0^+} \frac{1}{h} \psi(\lambda(E,(x,x+h))) > 0$.

Another definition is used in [31] and [32].

Definition 2.36. Let $g \in G$, $H \subset G$, $E \subset R$ and $x \in R$. We say that

(i) E is g-porous at x if there exists a sequence of intervals $\{I_n\}$ with $I_n \cap (E \cup \{x\}) = \emptyset$ so that $I_n \longrightarrow x$ and $\mathrm{dist}(x,I_n) < g(|I_n|)$. The right, left and bilateral g-porosities are defined similarly;

(ii) the set E is H-porous at x if there is $g \in H$ such that E is g-porous at x;

(iii) E is strongly H-porous at x if for every $g \in H$, E is g-porous at x.

A very similar definition is used in [66].

Definition 2.37. Let $E \subset R$, $x \in R$ and ψ be a porosity function. Then the right (ψ)-porosity index of E at x is defined as $PI_{\psi}^{+}(E,x) = \limsup_{\varepsilon \to 0^+} \left\{ \frac{\psi(k)}{h} : (x+h,x+h+k) \cap E = \emptyset,\ h+k < \varepsilon,\ 0 \leq h,\ 0 \leq k \right\}$. If this index is positive (is ∞), then E is said to be (ψ)-porous (or strongly porous) on the right at x.

Finally, the following definition is used in [92].

Definition 2.38. Let g be a porosity function. We say that a set $E \subset R$ is $[g]$-porous on the right at a point $x \in R$ if there is a sequence $h_n \searrow 0$ such that $g(\lambda(E,(x,x+h_n))) > h_n$.

Remark 2.39. If the porosity function equals to x^q, $0 < q < 1$, then all the above definitions give the same notion of corresponding generalized σ-porosity on R.

Moreover, the following theorem holds (for further generalizations see 4.B below).

Theorem 2.40 [82]. Let P be a metric space. Then the notion of σ-(x^q)-porosity in P does not depend on $0 < q < 1$.

The following proposition [82] is easy to prove.

Proposition 2.41. Let $0 < q < 1$. Then there exists a perfect (x^q)-porous set $E \subset R$ of positive Lebesgue measure.

Remark 2.42. (a) Definitions 2.34 and 2.36 (for the families $H = \{ax; a > 1\}$ and $H = \{ax; a > 0\}$) yield the notion of strong porosity.

(b) A generalization of Denjoy's Proposition 2.7 using strong bilateral (ψ)-porosity is contained in [66], p. 200.

(c) Some relationships between the above definitions are mentioned in [92].

For further results on generalized porosity see 4.B below.

2.E Globally and [g]-totally porous sets.

We start with some technical definitions.

Definition 2.43. If $I \subset R$ is an open interval and $r > 0$, then $J = r * I$ is the concentric interval with $|J| = r|I|$.

Definition 2.44. If $\{a_n\}_{-\infty}^{\infty}$ is a sequence with $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to-\infty} a_n = -\infty$ and $\cdots < a_{-2} < a_{-1} < a_0 < a_1 < \cdots$, we say that $D = \{[a_n, a_{n+1}] : n$ is an integer$\}$ is a division of R. If $a_{n+1} - a_n$ is constant, then D is called an equidistant division. The norm of D is defined as $\inf\{a_{n+1} - a_n\}$.

The following definition of globally porous sets was given in [38].

Definition 2.45. Let $E \subset R$ be a bounded set and $r > 0$. If E has at least two points, put $a = \inf E$, $b = \sup E$ and let $(a,b) - \overline{E} = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ are pairwise disjoint open (possibly empty) intervals. Let EP(N) denote the set of endpoints of the intervals $I_1, \ldots, I_{N-1}$. If there is an $r > 0$ such that for each N there is an N^* such that $E - EP(N) \subset \bigcup_{n=N}^{N^*} r * I_n$, then E is called r-globally porous. If card $E < 2$, then E is r-globally porous by definition. A set is called globally porous, if it is r-globally porous for some $r > 0$.

It is not difficult to prove that the following definition of a globally porous set is equivalent to the original one. For the definition of $\lambda(E,I)$ see Definition 2.1.

Definition 2.47. We say that a bounded set $E \subset R$ is globally porous, if there is $c > 0$ such that for any $d > 0$ there exists a division D of R with the norm less than d such that for each $I \in D$ the inequality $\lambda(E,I) > c|I|$ holds.

Clearly a globally porous set is porous. A remarkable property of globally porous sets is contained in the following lemma.

Lemma 2.47 [38]. If E is globally porous, then $\overline{E}$ is globally porous as well.

The proof of the following proposition [38] is based on Lemma 2.47.

Proposition 2.48. There is a perfect porous set $P \subset R$ which is not σ-globally porous.

The notion of a [g]-totally porous set from [92] is similar to the notion of a globally porous set.

Definition 2.49. Let g be a porosity function. We say that $E \subset R$ is a [g]-totally porous set if for each $d > 0$ there is an equidistant division D of R with the norm less than d such that $g(\lambda(E,I)) > |I|$ for each $I \in D$.

2.F Very porous sets.

Roughly speaking, if in a definition of a type of porosity "lim inf" is used instead of "lim sup", for the corresponding "porous sets" the name "very porous sets" will be used.

In the following, P is a metric space and $\gamma(x,M,R)$ is the number from Definition 2.22.

Definition 2.50. Let $E \subset P$, $x \in P$. Then we say that E is very porous at x if $\liminf_{R\to 0^+} \gamma(x,R,M)R^{-1} > 0$. We say that E is very strongly porous at x if $\lim_{R\to 0^+} 2\gamma(x,R,M)\cdot R^{-1} \geq 1$. The notions of very (strongly) porous sets and σ-very (strongly) porous sets are defined in the obvious way.

Definition 2.51. A set $M \subset P$ is called

(i) globally very porous if there exists $c > 0$ such that $\gamma(x,R,M) > cR$ for each $x \in P$ and $R > 0$, and

(ii) uniformly very porous if there exists $c > 0$ such that $\liminf_{R\to 0^+} \gamma(x,R,M)R^{-1} > c$ for each $x \in M$.

The above notions were considered in the theory of quasiconformal mappings in R^n ([61], [30], [69]).

The globally very porous sets were considered in [30] under the name "thin sets" and in [69] under the name "porous sets".

In [69] uniformly very porous sets were considered under the name "locally porous sets" and uniformly porous sets under the name "weakly locally porous sets".

Mattila considered very strongly porous sets in [48] and [49]; cf. 4.F.

2.G Superporous sets.

The superporosity was defined [86] in a connection with an investigation of the $\mathcal{I}$-density topology on R, which was considered by Wilczyński and others (cf. [72]).

In the following P is a metric space.

Definition 2.52. We say that $E \subset P$ is superporous at $x \in P$ if $E \cup F$ is porous at x whenever F is porous at x. A set $G \subset P$ is said to be p-open (porosity open) if $P - G$ is superporous at each point of G.

Remark 2.53. (a) The system of all sets which are superporous at x obviously forms an ideal. Therefore the system of all p-open sets forms a topology p which is called porosity topology.

(b) The system of all sets of the form $G = H - N$, where H is p-open and N is a first category set, forms a topology, called p^*-topology.

(c) In the case $P = R$ the p^*-topology coincides with the $\mathfrak{J}$-density topology [86], [90].

The superporosity is a rather restrictive notion. For example, Proposition 8 of [86] implies the following simple fact.

Remark 2.54. If $E \subset P$ is superporous at $x \in P$, then E is very porous at x.

If we consider strong porosity instead of ordinary porosity in above definitions, then we obtain notions of "strong p-topology" and "strong p^*-topology". Some of their properties were investigated by Kelar [40].

3. Applications.

3.A Cluster sets.

Let G denote a Jordan domain in R^n with smooth boundary Γ. For any $x \in \Gamma$, V^x denotes a Stolz angle in G (with vertex at x), and $C(x,V^x,f)$ denotes the corresponding Stolz cluster set of a (real or complex) function f defined in G. A point $x \in \Gamma$ is said to belong to $E_{VV}(f)$ if there exist two Stolz angles V_1^x, V_2^x such that $C(x,V_1^x,f) \neq C(x,V_2^x,f)$.

Dolženko [18] has proved the following theorem.

Theorem 3.1. For an arbitrary f on G, $E_{VV}(f)$ is a $G_{\delta\sigma}$ set which is a σ-porous subset of Γ.

The most interesting case is when $n = 2$ and G is the unit disc; other authors consider the (factually equivalent) case when G is the upper halfplane.

After the Dolženko's paper a number of articles investigating similar questions followed ([74], [75]-[79], [1], [81], [42], [19], [51], [70], [6], [46], [47]).

Yanagihara [74] considered cluster sets with respect to some "generalized angles" - regions which are bounded by two curves of a certain type. This investigation led him to the definition and an application of σ - (x^q)-porous sets $(0 < q < 1)$. These sets are applied also in [76].

Dolženko [18] has proved that Theorem 3.1 is in a sense the best possible. In fact, he proved that for each σ-porous set $E \subset \Gamma$ there exists a function f on G such that $E \subset E_{VV}(f)$.

It seems that the following question is still open even in the case when G is the unit disc.

__Question 3.2.__ Let $E \subset \Gamma$ be a σ-porous $G_{\delta\sigma}$ set. Does there exist f on G such that $E = E_{VV}(f)$?

Some partial answers to this question are given in [19] (the case of a σ-porous F_σ set), [70] and [42], where a characterization of sets $E_{VV}(f)$ is given.

The following theorem [81] is an example of a "one-dimensional" result in the theory of cluster sets, which applies σ-porosity. The symbols $C_e^+(f,x)$ and $C_e^-(f,x)$ denote the right and left essential cluster sets of f at x (cf. [4]).

__Theorem 3.3.__ If $f : R \longrightarrow R$ is arbitrary, then the set of points $x \in R$ at which $C_e^+(f,x) \neq C_e^-(f,x)$ is a σ-porous set of the type $F_{\sigma\delta\sigma}$.

__Remark 3.4.__ Samuels [60] has shown that the exceptional set from Theorem 3.3 can have the Hausdorff dimension 1.

__3.B Differentiation theory in R.__

These applications can be divided roughly into two groups — the first uses porosity and the second works with σ-porosity.

__Applications of porosity.__ In most applications of this type, the porosity computations are used in a definition of a generalized derivative or in assumptions of theorems. The first results of this sort are probably due to Denjoy (cf. [15] - Lemma 3, p. 106, Corollary 2, p. 109, Theorem 1, p. 139). A result of Khintchine [41] of this type is not correct (cf. [10]). Denjoy defined the notion of the index of a point w.r.t. a perfect set which is equivalent to the notion of the symmetric porosity (cf. 2.B) and applied it in the study of second symmetric Schwartz derivative and the corresponding T^2_S - totalization which has applications to trigonometric series. Porosity computations are used in many subsequent works (e.g. [20], [80], [62], [44]).

The best reference for recent theorems (e.g. [12], [64], [10], [11], [63]) is Thomson's book [66], pp. 26, 123-127, 153, 155-157, 159, 161, 165, 191-196, 203-205. One from these theorems is the following Thomson's result ([64, [66, p. 153]).

Theorem 3.5. Let S be a local system of sets that has the property that, for each $x \in R$ and for each set $M \in S(x)$, M is not bilaterally strongly porous at x. Then, for any continuous function $f : R \longrightarrow R$, the set of points

$$\{x : (S) - \overline{D}f(x) \neq \overline{D}f(x) \quad \text{or} \quad (S) - \underline{D}f(x) \neq \underline{D}f(x)\}$$

is of the first category.

Thomson [64] asked whether the assumptions concerning S can be weakened. By [92] it is sufficient to assume that there exists a porosity funtion g, $g(t) > t$, for which no $M \in S(x)$ is bilaterally [g]-porous at x. The following question [92] is open.

Question 3.6. Let S be a local system such that for any porosity function g, $g(t) > t$, there exists a second category set $M_g \subset R$ such that for any $x \in M_g$ there is a set $A \in S(x)$ which is bilaterally [g]-porous at x. Does there exist a continuous function f such that $\overline{D}f(x) \neq (S) - \overline{D}f(x)$ for all x from a second category set?

Applications of σ-porosity.

The first theorem in the differentiation theory which asserts that a set of "singular" points is σ-porous is due to Belna, Evans and Humke [5]. They proved the following theorem in which $C(f)$ denotes the set of all points at which f is continuous.

Theorem 3.7. If f is a function such that $C(f)$ is dense, then for all but a σ-porous set of points both of the following equalities hold:

(i) $D_s f(x) = \min\{D_+ f(x), D_- f(x)\}$

(ii) $D^s f(x) = \max\{D^+ f(x), D^- f(x)\}$.

Similar subsequent theorems concern Dini derivates of monotone and Lipschitz functions [23], Dini derivates of arbitrary functions [7], [83], approximate derivates [85], [55] and qualitative derivates [25]. All these theorems except Theorem 3.7 can be found in Thomson's book [66].

Note that σ-strongly porous sets were firstly applied in [83] (cf. [66], p. 177).

In some theorems both porosity and σ-porosity are applied, e.g. in Thomson's Theorem 66.3 of [66]. We present here its analogue proved in [12] (cf. [66], p. 161).

<u>Theorem 3.8</u>. Let ψ be a porosity function with $\psi'_+(0) = \infty$ and let f be a continuous function such that

$$|f(x) - f(y)| \leq \psi(|x - y|) \quad \text{whenever} \quad |x - y| \leq 1 .$$

Let S be a local system that has the property that, for each $x \in R$ and for each set $M \in S(x)$, M is not bilaterally strongly porous at x. Then at every point x, with the possible exception of a set that is σ - (ψ)-porous (in the sense of Definition 2.35 or 2.37),

$$(S) - \overline{D}f(x) = \overline{D}f(x) \quad \text{and} \quad (S) - \underline{D}f(x) = \underline{D}f(x) .$$

3.C <u>Typical behaviour</u>.

There is a number of theorems which concern level sets of typical continuous functions on [0,1] (cf. [8] and [65]) or, more generally, intersections of a typical continuous function with functions from a fixed classs of functions F. In [9] F is an arbitrary σ-compact class of functions, in [35] F is the class of all monotone functions and in [31] and [32] F is the class of all ϕ-Lipschitz (Hölder) functions given by a fixed "modulus of continuity" ϕ. In all these cases porosity and generalized porosity are natural tools for "measuring of smallness" of the intersection sets.

The case when F is the class of all absolutely continuous functions or the class of all functions of bounded variation was considered in [35], [13], and [14]. Buczolich proved the following proposition [13].

__Theorem 3.9__. Every continuous function on $[0,1]$ agrees with an absolutely continuous function on a set which is not bilaterally strongly porous.

Buczolich obtained results also in the opposite direction. For example, Theorem 1 of [14] easily implies the following

__Proposition 3.10__. Let $0 < \alpha < 1$. Then, for a typical continuous function f on $[0,1]$, the set $\{x : f(x) = g(x)\}$ is bilaterally (x^α)-porous for each function g of bounded variation.

__Remark 3.11__. (a) Any definition of (x^α)-porosity can be chosen.

(b) It seems that it is not known whether it is possible to write "strongly porous" or "bilaterally porous" instead of "bilaterally (x^α)-porous" in Proposition 3.10.

Theorems dealing with differential properties of typical continuous functions are very close to theorems dealing with intersection sets. They are contained, e.g., in [9], [31], [32], [92]. [73]; an application of both porosity and σ-porosity is quite natural in theorems of this type.

Zamfirescu [94] proved that all convex bodies in R^n, except those which belong to a σ-porous set, are smooth and strictly convex.

Larson [45] proved that a typical compact subset of $[0,1]$ is bilaterally strongly porous.

Note that at some cases an application of porosity notions is only one from possibilities how to describe that a set is very small. For example, a result of [92] which uses the notion of a $[g]$-totally porous set is only one from possible consequences of a theorem (see [91], p. 106) which uses the notion of $\mathfrak{N}$-game.

__3.D Banach spaces.__

Most applications of porosity in Banach spaces concern the differentiation theory. The following theorem is from [56].

__Theorem 3.12__. The set of points of Fréchet nondifferentiability of any continuous convex function on a Banach space with a separable dual is σ-porous.

This theorem was generalized in [87], where it is shown that the convexity can be replaced by a weaker property — the almost subdifferentiability.

Theorem 3.12 is improved in [57], where it is shown that the exceptional set from Theorem 3.12 is "σ-porous" in a more restrictive sense — it is "angle small". The question of sharpness of this result naturally led [57] to another type of "σ-porosity", the "ball smallness", which is more restrictive than angle smallness.

The angle smallness is applied to the differentiation of distance functions in Banach spaces in [89]. The notion of directionally porous sets (cf. Definition 2.32 above) was considered in the same context in [88], p. 299.

Agronsky and Bruckner [2] proved the following theorem which uses the notion of a totally porous set (cf. Definition 2.31).

Theorem 3.13. Let X be a separable Banach space and A a closed non-locally compact convex subset of X with more than one element. Let B be any compact subset of A. Then B is totally porous with respect to A.

4. Structural results on σ-porous sets.

4.A A generalization of a Foran's lemma.

Most proofs ([82], [67], [37], [26], [34]) which show that a perfect nowhere dense set is non σ-porous are based on a construction of a decreasing sequence of closed sets. Foran used in [26] a lemma which makes some proofs of this type more transparent. We present here an abstract reformulation of Foran's lemma, which is applicable to all types of σ-porosity.

Definition 4.1. Let P be a metric space and let $V = V(x,A)$ be a relation between points $x \in P$ and sets $A \subset P$. We shall say that V is a porosity relation if:

(i) If $A \subset B$ and $V(x,B)$, then $V(x,A)$;

(ii) $V(x,M)$ iff there is $r > 0$ such that $V(x, M \cap B(x,r))$;

(iii) $V(x,A)$ iff $V(x,\bar{A})$.

We say that A is V-porous at x if $V(x,A)$ holds. The notions of V-porous and σ-V-porous sets are defined in the obvious way.

Definition 4.2. Let P be a complete metric space and V a porosity relation. We say that $\mathcal{F} \subset \exp P$ is a non-σ-V-porosity family if the following conditions hold:

(a) $\mathcal{F}$ is a nonempty family of nonempty closed sets.

(b) For each $F \in \mathcal{F}$ and each open set $G \subset P$ with $F \cap G \neq \emptyset$ there exists $F^* \in \mathcal{F}$ such that $\emptyset \neq F^* \cap G \subset F \cap G$ and $F \cap G$ is V-porous at no point of $F^* \cap G$.

Lemma 4.3. Let $\mathcal{F}$ be a non-σ-V-porosity family in a complete metric space. Then no set from $\mathcal{F}$ is σ-V-porous.

Proof. Suppose on the contrary that $F \in \mathcal{F}$ is σ-V-porous. Then $F = \bigcup_{n=1}^{\infty} A_n$, where each A_n is a V-porous set. We shall define inductively a sequence $\{F_n\}_{n=0}^{\infty} \subset \mathcal{F}$ and a sequence of open balls $\{B(x_n,r_n)\}_{n=0}^{\infty}$ such that $r_n \to 0$, $F = F_0$ and $F_{n-1} \cap \overline{B(x_{n-1}, r_{n-1})} \supset F_n \cap \overline{B(x_n,r_n)}$, $F_{n-1} \cap B(x_{n-1}, r_{n-1}) \neq \emptyset$, $F_n \cap \overline{B(x_n,r_n)} \cap A_n = \emptyset$ for $n = 1,2,\dots$:

1. Put $F_0 = F$, choose $x_0 \in F$ and put $r_0 = 1$.
2. If F_{n-1} and $B(x_{n-1}, r_{n-1})$ are defined for some $n \geq 1$, then we define F_n and $B(x_n,r_n)$ distinguishing two cases:

   2a. If $A_n \cap F_{n-1} \cap B(x_{n-1}, r_{n-1})$ is not dense in $F_{n-1} \cap B(x_{n-1},r_{n-1})$, then put $F_n = F_{n-1}$ and choose $x_n \in F_{n-1} \cap B(x_{n-1},r_{n-1})$ and $r_n < 1/n$ such that $F_{n-1} \cap \overline{B(x_n,r_n)} \cap A_n = \emptyset$.

   2b. If $A_n \cap F_{n-1} \cap B(x_{n-1},r_{n-1})$ is dense in $F_{n-1} \cap B(x_{n-1},r_{n-1})$, then we use Definition 4.2(b) (for $F = F_{n-1}$ and $G = B(x_{n-1},r_{n-1})$) and obtain a set $F_n \in \mathcal{F}$ such that $F_{n-1} \cap B(x_{n-1},r_{n-1})$ is V-porous at no point of $F_n \cap B(x_{n-1},r_{n-1}) \neq \emptyset$. Since $F_{n-1} \cap B(x_{n-1},r_{n-1}) \subset \overline{A}_n$ and A_n is a V-porous set, we obtain by Definition 4.1 that $F_{n-1} \cap B(x_{n-1},r_{n-1})$ is V-porous at each point of A_n and consequently $A_n \cap F_n \cap B(x_{n-1},r_{n-1}) = \emptyset$. Now choose $x_n \in F_n$ and $0 < r_n < 1/n$ such that $\overline{B(x_n,r_n)} \subset B(x_{n-1},r_{n-1})$.

Clearly $\bigcap_{n=1}^{\infty} F_n \cap \overline{B(x_n,r_n)} = \{a\}$, $a \in F$ and $a \notin \bigcup_{n=1}^{\infty} A_n = F$, which is a contradiction.

4.B Some results on generalized σ-porosity.

We shall reproduce some results of [82]. The main results show that under some assumptions a notion of generalized σ-porosity coincides with another one.

In the sequel G denotes the system of all porosity functions and σ-(g)-porosity is considered in the sense of Definition 2.33.

Proposition 4.4. Let $h \in G$, $f \in G$ be porosity functions with the following property: there is an integer r and $d > 0$ so that the r-fold composition

$$h \circ h \circ \cdots \circ h(x) \geq f(x) \quad \text{for} \quad 0 < x < d .$$

Then, in an arbitrary metric space, any set that is σ-$\langle f \rangle$-porous is necessarily also σ-$\langle h \rangle$-porous.

As an easy consequence we obtain the following

Proposition 4.5. Let h and f be porosity functions with the following property: for any $B > 0$ there are $A > 0$, $d > 0$, and an integer r so that the r-fold composition

$$(Ah) \circ (Ah) \circ \cdots \circ (Ah)(x) \geq Bf(x) \quad \text{for} \quad 0 < x < d .$$

Then, in an arbitrary metric space, any set that is σ-(f)-porous is necessarily also σ-(h)-porous.

Remark 4.6. (a) Proposition 4.5 for subsets of R is proved in [66] for σ-(g)-porosity in the sense of Definition 2.37.

(b) Proposition 2.15 and Theorem 2.40 above are the most important consequences of Propositions 4.4 and 4.5.

In the sequel $\tilde{G}$ denotes the system of all functions $g \in G$ for which $\infty > g'(x) > 1$. The following proposition shows that the assumptions of Proposition 4.4 cannot be essentially weakened.

Proposition 4.7. Let $f \in G$ and $H \subset \tilde{G}$. Let there exist a sequence $\{h_i\} \subset H$ and a sequence $\{d_n\}$ of positive numbers such that

$$h_n \circ \cdots \circ h_1(x) < f(x) \quad \text{for} \quad 0 < x < d_n .$$

Then in any Euclidean space there exists an $\langle f\rangle$-porous perfect set F which is not σ-$\langle H\rangle$-porous.

Remark 4.8. Proposition 4.7 implies (via Remark 2.42) the existence of a porous set which is not σ-strongly porous. Of course, Theorem 2.14 implies that the ternary Cantor set has this property. Proposition 4.7 also implies a partial converse to Proposition 4.5 (see Proposition 5.3 of [82]).

Proposition 4.9. Let $g \in G$ and $\lim_{x\to 0^+} x/g(x) = 0$. Then in any Euclidean space there exists a perfect set of measure zero which is (g)-porous and is not σ-porous.

Proposition 4.7 implies that, for each $g \in G$, there exists a perfect nowhere dense $P \subset R$ which is not σ-(g)-porous. The question whether P can be chosen to be a measure-zero set was asked in [82] and [33]; also the case $g(x) = x^q$, $0 < q < 1$, was open. The positive answer to this question was given by Konjagin in 1985; see Theorem 5.2 below.

4.C Non σ-porous sets need not be big

Let $A \subset R$ be a Borel set which is not of the first category or is not of measure zero. Then A is in a sense big, e.g. $A + A$ contains an interval (Steinhaus theorem). Further, each family of pairwise disjoint sets of the same type as A is clearly countable.

For the σ-ideal of σ-porous sets no such results hold. Tkadlec, answering questions asked by Humke [33] and Wilczyński, proved the following results [67].

Theorem 4.10. There exists a perfect non σ-porous set $S \subset R$ such that for every finite sequence $(c_1,\ldots,c_n) \in R^n$ the set $\sum_{j=1}^{n} c_j S$ is of measure zero.

Theorem 4.11. Let $K \subset R$ be a first category set. Then there exists a perfect non σ-porous set S of measure zero disjoint from K. Consequently there exists an uncountable family of pairwise disjoint non σ-porous perfect subsets of R.

Remark 4.12. There is a perfect porous set P (e.g. the Cantor ternary set) such that $P - P$ contains an interval. Moreover, such set P can be found in each second category set with the Baire property [50].

Reclaw [58] showed that Theorem 4.11 and a general result of Mycielski easily implies the following

Theorem 4.13. There exists a family of cardinality of the continuum of pairwise disjoint, non σ-porous, perfect subsets of R.

Foran [26] constructed a non σ-porous set using expansions (slightly more complicated than the decadic ones) of real numbers. A slight change of his construction gives a quite explicit example [93] of a family from Theorem 4.13.

Let $x \in (0,1)$. As usual, we write $x = 0,a_1,a_2,\ldots,$ if $x = \sum_{i=1}^{\infty} a_i(x) \cdot 10^{-i}$. The uniqueness of the expansion is obtained using terminating 0's whenever x has two expansions. Let $a \in \{0,1,\ldots,9\}$ be a digit. The density of a in the expansion of x is defined as

$$d(a,x) = \lim_{n\to\infty} \frac{\#\{k : 1 \leq k \leq n,\ a_k(x) = a\}}{n} .$$

Proposition 4.14. Let $a \in \{1,\ldots,8\}$. Then, for any $0 < d < 1$, the set $A_d = \{x : d(a,x) = d\}$ is a Borel non σ-porous set.

4.D Descriptive properties.

Reclaw [58] observed that each γ-set $X \subset R$ is σ-porous and, using a known result on γ-sets, obtained the following proposition concerning cardinality and σ-porosity.

Proposition 4.15. Assume Martin's axiom. Then every set $X \subset R$ of cardinality less than that of the continuum is σ-porous.

Foran and Humke obtained [29] the following results concerning "envelope properties" of linear sets.

Proposition 4.16. A. Every σ-porous set is contained in a $G_{\delta\sigma}$ σ-porous set.

B. Some σ-porous sets are contained in no G_δ first category set.

C. Some σ-porous sets are contained in no F_σ measure-zero set.

The question whether there exists a porous set which is contained in no σ-porous G_δ set was posed in [29]. It was answered by Tkadlec [67], p. 476:

Proposition 4.17. There exists a perfect nowhere dense set $S \subset R$ such that the set $P(S)$ of all points $x \in S$ at which S is porous is contained in no σ-porous G_δ set.

Remark 4.18. (a) $P(S)$ is clearly porous.

(b) It would be interesting to know whether an arbitrary non σ-porous perfect nowhere dense set has the property of S from Proposition 4.17.

(c) It seems that the question whether each porous (or σ-porous) set is contained in a σ-porous $F_{\sigma\delta}$ set was not considered in the literature.

Using Proposition 4.14 and the Lebesgue method of universal funtions it is not difficult to prove the following theorem [93] which shows a further difference between the σ-ideal of measure-zero, first category sets and the σ-ideal of σ-porous sets. This result answers a question mentioned by Wilczyński [71].

Theorem 4.19. Let B_α be the system of all linear Borel sets of additive class α. Then there exists a Borel set D such that, for each $C \in B_\alpha$, the symmetric difference $D \Delta C$ is not σ-porous.

We finish this section with a natural question which seems to be open.

Question 4.20. Let B be a non σ-porous Borel set. Does there exist a closed non σ-porous set $F \subset B$?

4.E Images of σ-porous sets.

It is easy to see that σ-porous sets (e.g. on R) are not invariant with respect to all homeomorphisms. On the other hand, we have the following obvious but useful proposition.

Proposition 4.21. Let P be a metric space and $f : P \to P$ a Lipschitz bijection such that f^{-1} is also Lipschitz. Then any set $A \subset P$ is porous (or σ-porous) iff $f(A)$ is.

A further related one-dimensional discussion is contained in [52].

Further results in another direction were obtained by Reclaw [58]. He observed that each γ-set is σ-porous, and, as consequences of known results on γ-sets, he obtained the following propositions.

__Proposition 4.22.__ Assume Martin's axiom. Then there exists $X \subset R$ of cardinality of the continuum such that every continuous image of X is σ-porous.

__Proposition 4.23.__ Assume that it is consistent that there exists a measurable cardinal. Then it is consistent that there exists $X \subset R$ of cardinality of the continuum such that every Borel image of X is σ-porous.

__4.F Measures and σ-porosity.__

Each σ-porous set $A \subset R^n$ is of Lebesgue measure zero. On the other hand, there exist (cf. Remark 3.4) σ-porous sets $A \subset R$ of Hausdorff dimension 1. Moreover, there exists [28] a bilaterally strongly porous perfect set $A \subset R$ of Hausdorff dimension 1.

The connection between generalized porosity and Hausdorff measures was considered in [32]. If $h \in G$ is a porosity function, then we denote by μ^h the Hausdorff measure determined by h. If $h(x) = x^\alpha$, then μ^h is the α-dimensional Hausdorff measure H^α. The following theorem is proved in [32] (cf. also [66], p. 202).

__Proposition 4.24.__ Let g be a porosity function and let $h = g^{-1}$. If $E \subset R$ is g-porous in the sense of Definition 2.36, then $\mu^h(E) = 0$.

For very porous sets the following results are known [48].

__Proposition 4.25.__ Let $E \subset R^n$, $n \geq 2$, $0 < p < 1$. If $\liminf_{r\to 0^+} 2\gamma(x,E,r)/r \geq p$ for $x \in E$, then the Hausdorff dimension $\dim E \leq d(p)$, where $d(p)$, $n - 1 \leq d(p) < n$, is a constant depending only on n and p.

__Remark 4.26.__ Proposition 4.25 shows that a uniformly very porous set $E \subset R^n$, $n > 1$, is of the Hausdorff dimension $\dim E < n$. This result for globally very porous sets is contained in [61].

Mattila [49] (cf. [48]) proved the following deeper result.

Theorem 4.27. $\lim_{p\to 1^-} d(p) = n - 1$. In particular, if E is very strongly porous, then $\dim E \leq n - 1$.

We shall say that a measure μ is absolutely continuous w.r.t. a σ-ideal I, if $\mu A = 0$ for each $A \in I$. Obviously, if a Radon measure on R is absolutely continuous w.r.t. the Lebesgue measure, then it is absolutely continuous w.r.t. the system of all σ-porous sets. Tkadlec [68] showed that the converse does not hold; he constructed a singular Radon measure μ in R such that each σ-porous sets is μ-null (we shall label such measures as T-measures).

This T-measure μ is concentrated on a perfect set of Lebesgue measure zero. Thus the construction of Tkadlec gives a "measure" proof of the existence of a perfect measure-zero non σ-porous set, unlike the other proofs. Tkadlec used the following interesting lemma which was formulated by D. Preiss (cf. [68], p. 353). The symbol $d * I$ is defined in Definition 2.43.

Lemma 4.28. Let μ be a finite Borel measure concentrated on $S \subset R$ and let the following conditions hold.

(1) There is $d > 1$ such that $\sum \mu(d * I) < \infty$, where I runs through the set of all bounded intervals contiguous to $\bar{S}$.

(2) There are $c > 1$, $K > 0$ and $d > 0$ such that $\mu(c * I) \leq K\mu(I)$ for every interval I with $|I| < d$ and with center in S.

(3) Countable sets are μ-null sets.

Then $\mu(P) = 0$ for every σ-porous set P.

Humke and Preiss [36] showed that, surprisingly enough, there are perfect non σ-porous linear sets which carry no T-measure.

In fact, they proved the following deep result.

Theorem 4.29. A symmetric perfect set $C(\alpha)$ (cf. Definition 2.11) carries a nontrivial Radon measure which is absolutely continuous w.r.t. the system of all σ-porous sets iff there exists $s > 0$ such that $\sum_{n=1}^{\infty} (\alpha_n)^s < \infty$.

Remark 4.30. (a) If $\alpha_n = (n+1)^{-1}$, then Theorem 4.29 gives a "measure" proof of non σ-porosity of $C(\alpha)$. If $\alpha_n = (\ln(n + 2))^{-1}$, then such "measure" proof does not exist.

(b) The definition of the notion of a Tkadlec measure in [36] is not convenient.

It would be of some interest to know relations between the systems of all Radon measures which are absolutely continuous w.r.t. the σ-ideals of σ-porous sets, σ-strongly porous sets, σ-bilaterally porous sets, σ-globally porous sets, etc. In particular, probably no answer to the following natural question was published.

__Question 4.31__. Does there exist a Radon measure in R which is absolutely continuous w.r.t. σ-strongly porous sets and is not absolutely continuous w.r.t. σ-porous sets?

For a recent result concerning T-measures in R^2 which are carried by a graph of a continuous function see 6.B below.

5. Konjagin's example.

In [43], Konjagin proved the following theorem. His construction gives, among other things, the shortest description of a closed non σ-porous set of measure zero.

__Theorem 5.1__. The set $E = \left\{ x : \sum_{n=1}^{\infty} |\sin(n!\pi x)|/n \leq 1 \right\}$ is a closed non σ-porous set of measure zero. Moreover, there exists a symmetrically continuous function on R which is discontinuous at every point of E.

Of course, the basic part of the proof of Theorem 5.1 is the proof of non σ-porosity of E. A well known fact from the theory of trigonometric series immediately implies that E is of measure zero. In fact, E is an N-set ($A \subset R$ is an N-set iff there are b_n such that $\sum |b_n| = \infty$ and $\sum |b_n \sin(\pi nx)| < \infty$ for each $x \in A$) and every N-set is of measure zero (cf. [3]). Since the sum from the definition of E is a lower semicontinuous function, E is closed. The last assertion of Theorem 5.1 follows from a construction of Preiss [53], who factually proved that for each N-set A there exists a symmetrically continuous function which is discontinuous at every point of A. Further note that Theorem 5.1 answers in negative the question [33] whether each N-set is σ-porous.

Konjagin also observed that a slight modification of the construction from Theorem 5.1 gives the following Theorem 5.2. We shall give here a proof of Theorem 5.2 only, since the proof of Theorem 5.1 is simpler and quite similar. All ideas of the proof are due to Konjagin, but he did not use Lemma 4.3.

Theorem 5.2. Let $g \in G$ be a porosity function. Then there exists a closed measure-zero set which is not σ-[g]-porous.

Proof. Define inductively a sequence $\{p_n\}_{n=1}^{\infty}$ of natural numbers so that

(1) p_{n+1}/p_n is a natural number ≥ 2, $n = 1,2,\ldots$ and

(2) $p_k\, g(1/p_n) < 1/\pi n^2$, $k = 1,\ldots,n-1$, $n = 1,2,\ldots$.

Put $E_a = \left\{ x : \sum_{k=1}^{\infty} (|\sin(p_k \pi x)|/k) \leq a \right\}$, $a > 0$. All E_a are closed measure-zero sets (cf. text after Theorem 5.1). We shall say that $a > 0$ is "good", if

(3) $\left\{ x : \sum_{k=1}^{\infty} (|\sin(p_k \pi x)|/k) < a \right\} = \bigcup_{0<b<a} E_b$ is dense in E_a.

Now observe that the set W of all $a > 0$ which are not "good" is countable. In fact, to each $a \in W$ corresponds an interval I_a with rational endpoints such that $I_a \cap E_a \neq \emptyset$ and $I_a \cap E_b = \emptyset$ whenever $b < a$. This property immediately implies that the mapping $a \longrightarrow I_a$ is injective and consequently W is countable. To prove the theorem, it is sufficient to prove that the system $\{E_a : a \in (0,\infty) - W\}$ is non-σ-[g]-porosity family in the sense of Definition 4.2. At first observe that $0 \in E_a$, $a > 0$. Further choose $a > 0$, $a \notin W$ and an open set G such that $E_a \cap G \neq \emptyset$. Since a is "good" and W is countable, we can choose $0 < b < a$ such that $b \notin W$ and $E_b \cap G$ is [g]-porous at no point of $E_b \cap G$. Suppose on the contrary that there exists $y \in E_b \cap G$ at which E_a is [g]-porous. Then for each $\varepsilon > 0$ we can find $0 < R < \varepsilon$ and an interval $(y_1,y_2) \subset (y-R, y+R) - E_a$ such that $g(y_2 - y_1) > R$. Fix $\varepsilon \leq 1/p$, R, y_1, y_2 and find n such that

$$(4) \qquad 1/p_{n+1} < y_2 - y_1 \leq 1/p_n \,.$$

Since $|p_{n+1} y_2 - p_{n+1} y_1| > 1$, we can find $z \in (y_1, y_2)$ such that $p_{n+1} z$ is an integer. On account of this fact and (1) we have

$$A := \sum_{k=1}^{\infty} (|\sin(p_k \pi z)|/k) = \sum_{k=1}^{n} (|\sin(p_k \pi z)|/k) \leq$$

$$\leq \sum_{k=1}^{n-1} (|\sin(p_k \pi z)|/k) + 1/n .$$

Further

$$\sum_{k=1}^{n-1} (|\sin(p_k \pi z)/k) \leq \sum_{k=1}^{n-1} (|\sin(p_k \pi y)|/k) +$$

$$+ \left| \sum_{k=1}^{n-1} (1/k)(|\sin(p_k \pi z)| - |\sin(p_k \pi y)|)\right| \leq$$

$$\leq b + \sum_{k=1}^{n-1} (1/k)p_k \pi |z - y| := b + B .$$

Since $|z - y| < R$, $g(y_2 - y_1) > R$ and $y_2 - y_1 \leq 1/p_n$, we obtain $g(1/p_n) > z - y$. On account of (2) we have $B \leq \sum_{k=1}^{n-1} (1/n^2) < 1/n$ and $A \leq b + 2/n$. On account of (1) and (4) we see that $A \leq b + 2/n \leq a$, if $\varepsilon > 0$ is sufficiently small. This implies $z \in E_a$, which is a contradiction.

<u>Remark 5.3</u>. It is easy to see that for each porosity function h there exists a porosity function g such that each (h)-porous set $A \subset R$ (in the sense of Definition 2.33 or in another one) is [g]-porous (cf. [92]). Consequently Theorem 5.2 holds for σ-(g)-porosity as well. Thus Theorem 5.2 solves problems from [82] and [33], which are mentioned after Proposition 4.9.

6. Other results.

6.A Graphs of continuous functions.

Foran [26] proved the following theorem.

<u>Theorem 6.1</u>. There is a continuous real function defined on [0,1] whose graph is a non σ-porous subset of the plane.

The following stronger result was recently proved by Bandt, Mattila and Preiss [54]. Let λ/M denote the restriction of the Lebesgue measure to a measurable set $M \subset R$ and let $G_f(x) = (x,f(x))$ for each $f : R \to R$.

Theorem 6.2. There is a continuous real function f on $[0,1]$ such that the image measure $G_f(\lambda/M)$ is absolutely continuous w.r.t. the system of all σ-porous subsets of R^2 (cf. 4.F) for each $M \subset [0,1]$, $\lambda M > 0$.

Remark 6.3. For the function f from Theorem 6.2, $\{(x,f(x)) : x \in M\}$ is non σ-porous for each $M \subset [0,1]$, $\lambda M > 0$.

It seems that the following question [26] is still open.

Question 6.4. Does there exist a function of bounded variation on $[0,1]$ which has non σ-porous graph?

Foran [27] proved the following

Proposition 6.5. There is an absolutely continuous function f on $[0,1]$ such that the set of points of non-porosity of the graph G_f is c-dense in G_f.

6.B Trigonometric series and porosity.

1. Denjoy used porosity computations (cf. 2.B) in his theory of T_s^2-totalization (cf. [15]) which was motivated by the theory of trigonometric series.

2. A result of Piatecki-Shapiro says that there exists a U-set which is not H_σ-set (for definitions of U-sets and H-sets see [3] or [39]). In [3] a proof of this result is given which uses the notion of porosity (under another nomenclature). The result is proved in three steps:

(i) It is observed that the closure of a H-set is bilaterally porous. (Note that it is easy to see that each H-set is globally porous).

(ii) A perfect U-set P is constructed which contains a dense subset $D \subset P$ such that P is non-porous on the right at each point of D.

(iii) The Baire category theorem easily yields that P is not an H_σ-set.

Remark 6.6. The above U-set P is σ-porous.

The following problem seems to be open.

Question 6.7. Is each Borel (perfect) U-set σ-porous?

Remark 6.8. (a) The negative answer is very probable.

(b) There are symmetric perfect porous sets which are not U-sets [3].

The question whether each N-set is σ-porous was asked by Humke in [33] and answered in negative by Konjagin (Theorem 5.1 above).

6.C Additional remarks.

A simple but interesting application of Theorem 3.7 which concerns approximately symmetric functions is contained in [24].

Some questions concerning σ-porosity of some exceptional sets are contained in [21]. In particular, the following problem [21] seems to be still open.

Question 6.9. Must the set of points of discontinuity of a symmetric function be σ-porous?

Note that the similar question concerning symmetrically continuous functions has the negative answer (see Theorem 5.1 above).

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