QUERIES

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THE SYMMETRIC DERIVATIVE AND THE DARBOUX PROPERTY

In connection with his survey paper [L1] Lee Larson has raised the following problem (Query 172): Find a condition which is both necessary and sufficient to ensure that a finite symmetric derivative is a Darboux-Baire one function. the aim of the present paper is to give another formulation of the above problem for locally bounded symmetric derivatives and to give a sufficient condition for a symmetric derivative to be a Darboux-Baire one function. Note that another sufficient condition is given in [L3].

We shall deal with mappings of the real line $\mathbb{R}$ into itself. The upper symmetric derivative of f at $x \in \mathbb{R}$ is

$$\overline{f}^s(x) = \limsup_{h\to 0} (f(x + h) - f(x - h))/2h$$

and the lower symmetric derivative $\underline{f}^s(x)$ is the corresponding limit inferior. When $\overline{f}^s(x) = \underline{f}^s(x)$, the common value is denoted by $f^s(x)$ and is called the symmetric derivative of f at x. It is known that every symmetric derivative belongs to the first Baire class ([L2], Th. 2.1). Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to have the Darboux property if whenever $a,b \in \mathbb{R}$, $a < b$, and y is any number between $f(a)$ and $f(b)$, there is a number $z \in (a,b)$ such that $f(z) = y$.

In the paper [E] the class M_{-1} of functions with some pleasant properties is defined.

<u>Definition 1</u>. A function $f : \mathbb{R} \to \mathbb{R}$ is in M_{-1} if it is measurable and for each x, $\liminf_{t\to x} f(t) \le f(x) \le \limsup_{t\to x} f(t)$.

<u>Theorem E</u>. ([E], Th. 2, quasi-mean value theorem) Let $f : \mathbb{R} \to \mathbb{R}$ be in M_{-1}. If $a < b$, then there exist $x_1,x_2 \in [a,b]$ such that $\underline{f}^s(x_1) \le (f(b) - f(a))/(b - a) \le \overline{f}^s(x_2)$.

Further we shall deal with locally bounded symmetric derivatives. Every such symmetric derivative is bounded on every compact interval.

Definition 2. Let $f : \mathbb{R} \to \mathbb{R}$ be symmetrically differentiable. The function f is said to belong to class MVT (of functions fulfilling the mean value theorem) if for any $a,b \in \mathbb{R}$, $a < b$, there exists $z \in [a,b]$ such that $f(b) - f(a) = f^s(z)(b - a)$, and, whenever for some $x \in \mathbb{R}$ $\lim_{t\to x+} f_s(t)$ ($\lim_{t\to x-} f^s(t)$) exists, it equals $f^s(x)$.

Theorem. Let h be a locally bounded symmetric derivative. Then h has the Darboux property if and only if there exists a function, $w \in$ MVT with $\mu^s = h$.

Proof. Let $H : \mathbb{R} \to \mathbb{R}$ be such that $H^s = h$. Then H is obviously symmetrically continuous, i.e. $\lim_{k\to 0} (H(x + k) - H(x - k)) = 0$ for each $x \in \mathbb{R}$. According to the paper [B] almost all points of $\mathbb{R}$ belong to the set $C(H)$ of all its points of continuity and hence H is measurable. Let $[u,v] \subset \mathbb{R}$ be a compact interval. Suppose that for each $x \in [u,v]$ we have $|h(x)| < c$ $(c > 0)$. Choose $\alpha \in C(H) \cap (u,v)$. If $\beta \in C(H) \cap (u,v)$, $\alpha < \beta$, then $h(p) \leqslant (H(\beta) - H(\alpha))/(\beta - \alpha) \leqslant h(q)$ for some $p,q \in (\alpha,\beta)$ ([L2], Th. 7.1). Since $|h(p)| < c$ and $|h(q)| < c$, we have $|H(\beta)| \leqslant |H(\alpha)| + c|\beta - \alpha|$. The same estimate holds for $\beta < \alpha$. Consequently H is bounded on the set $C(H) \cap (u,v)$.

In [L2] it is shown that for a given symmetric derivative h there is a Baire one function $\mu : \mathbb{R} \to \mathbb{R}$, $\mu = \mu_H$, which is a primitive for h (i.e., $\mu^s = h$). Since H is bounded on $C(H) \cap (u,v)$, the function μ is determined for $x \in (u,v)$ by

$$\mu(x) = \limsup_{\substack{t\to x \\ t\in C(H)}} H(t)$$

and $\mu^s(x) = H^s(x) = h(x)$ holds for each $x \in (u,v)$. Obviously $\liminf_{t\to x} \mu(t) \leqslant \mu(x) \leqslant \limsup_{t\to x} \mu(t)$ is fulfilled for every $x \in (u,v)$. Since $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n,n]$, we can suppose that $\mu \in M_{-1}$ and $\mu^s(x) = h(x)$ holds for each $x \in \mathbb{R}$. The function μ fulfills the assumptions of Theorem E, according to which for each $a,b \in \mathbb{R}$, $a < b$, there are $x_1,x_2 \in [a,b]$ such that

(*) $h(x_1) \le (\mu(b) - \mu(a))/(b - a) \le h(x_2)$.

Suppose that h has the Darboux property. It follows immediately from (*) that $\mu(b) - \mu(a) = h(z)(b - a) = \mu^s(z)(b - a)$ for some $z \in [a,b]$. If for some $x \in \mathbb{R}$, $\lim_{t\to x+} h(t)$ (or $\lim_{t\to x-} h(t)$) exists, then it equals $h(x)$. This follows from Young's characterization of Darboux Baire class 1 functions. (See [Br], p. 9.) Hence $\mu \in MVT$.

Now suppose that there is a function $\mu \in MVT$ with $\mu^s = h$ and that h is not a function with the Darboux property. Then there are $a,b \in \mathbb{R}$, $a < b$, and a number y between $h(a)$ and $h(b)$, such that $h(x) \neq y$ holds for each $x \in (a,b)$. We shall treat the following two possibilities: a) there are $x_1,x_2 \in (a,b)$ such that $h(x_1) < y < h(x_2)$; b) $h(x) > y$ (or $h(x) < y$) holds for each $x \in (a,b)$.

a) Let $[u,v]$ be a compact interval, $u < a < b < v$ and let $|h(x)| < c$ $(c > 0)$ hold for every $x \in (u,v)$. Since $\mu \in MVT$, $|\mu(x) - \mu(x_0)| < c|x - x_0|$ for every $x,x_0 \in (u,v)$. Hence μ is a continuous function on (u,v). Put $\varphi_k(x) = (\mu(x + k) - \mu(x - k))/2k$ for $k > 0$ and $x \in [a,b]$. We can suppose, without loss of generality, that $x_1 < x_2$. For sufficiently small $k > 0$ we have $a < x_1 - k < x_2 + k < b$ and $\varphi_k(x_1) < y < \varphi_k(x_2)$. Since φ_k is a continuous function, there exists a suitable r, $0 < r \le k$, such that $\varphi_k(x_1 + r) < y < \varphi_k(x_2 - r)$. Since φ_k has the Darboux property, for some $w \in (x_1 + r,x_2 - r)$ we have $\varphi_k(w) = y$, i.e. $\mu(w + k) - \mu(w - k) = 2ky$. It follows from $\mu \in MVT$ that there exists a point $z \in [w - k,w + k]$ such that $\mu(w + k) - \mu(w - k) = 2k\,\mu^s(z)$. Consequently $y = h(z)$ and $z \in (x_1 - (k - r),x_2 + (k - r)) \subset (a,b)$ contrary to assumption.

b) Let $h(a) < y < h(b)$. Suppose that $h(x) > y$ holds for each $x \in (a,b)$. We may also suppose $y = 0$. (This can be seen by inspecting the function $\nu(x) = \mu(x) - yx$.) According to Theorem 6.2 of [L2] the function μ is continuous and nondecreasing on (a,b). Hence $\lim_{t\to a+} h(t) \ge y > h(a)$ exists contrary to assumption. The case $h(x) < y$ for each $x \in (a,b)$ can be treated analogously.

In [M] the following analogy of Lagrange's mean value theorem is proved. ($D^+f(x)$, $D^-f(x)$, $D_+f(x)$, $D_-f(x)$ denote the upper right, upper left, lower right and lower left derivates of f at x.)

Theorem M. Let f be continuous in $[a,b]$ and symmetrically differentiable in (a,b). Let

$$E = \{x \in [a,b] : f(x) - (f(b) - f(a))x/(b - a) > (bf(a) - af(b))/(b - a)\},$$

$$F = \{x \in [a,b] : f(x) - (f(b) - f(a))x/(b - a) < (bf(a) - af(b))/(b - a)\}.$$

Suppose that $D^+f(x) \geq D_-f(x)$ for $x \in E$ and $D_+f(x) \leq D^-f(x)$ for $x \in F$. Then there exists at least one point ξ in (a,b) such that $f(b) - f(a) = (b - a)f^s(\xi)$.

Corollary. Let $h : \mathbb{R} \to \mathbb{R}$ be a locally bounded symmetric derivative such that whenever $\lim_{t\to x+} h(t)$ $(\lim_{t\to x-} h(t))$ exists, it equals $h(x)$. Let μ be a continuous primitive of h (i.e. $\mu^s = h$). If for any $a,b \in \mathbb{R}$, $a < b$, we have $D^+\mu(x) \geq D_-\mu(x)$ for $x \in E$ and $D_+\mu(x) \leq D^-\mu(x)$ for $x \in F$, where

$$E = \{x \in [a,b] : \mu(x) - (\mu(b) - \mu(a))x/(b - a) > (b\mu(a) - a\mu(b))/(b - a)\},$$

$$F = \{x \in [a,b] : \mu(x) - (\mu(b) - \mu(a))x/(b - a) < (b\mu(a) - a\mu(b))/(b - a)\}.$$

then h has the Darboux property.

Proof. It follows from (*) of the first part of the proof of the Theorem that for any locally bounded symmetric derivative h there exists a continuous primitive μ. The statement of the Corollary is an immediate consequence of the Theorem and Theorem M.

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