Ján Borsík, Matematický ústav SAV, Ždanovova 6, 040 01 Košice, Czechoslovakia

Jozef Doboš, Katedra matematiky VŠT, Švermova 9, 040 01 Košice, Czechoslovakia

ON METRIC PRESERVING FUNCTIONS

Definition 1. We call a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ metric preserving iff $f \circ d : M \times M \longrightarrow \mathbb{R}^+$ is a metric for every metric $d : M \times M \longrightarrow \mathbb{R}^+$, where (M,d) is an arbitrary metric space and $\mathbb{R}^+$ denotes the set of nonnegative reals. We denote by $\mathfrak{M}$ the set of all metric preserving functions.

In the papers [1] and [2] some properties of metric preserving functions were investigated. The purpose of this paper is to extend some results of [1] and [2]. We shall show that each metric preserving function has a derivative (finite or infinite) at 0.

We recall some properties of $\mathfrak{M}$.

Proposition 1. (See [1; Theorem 1].) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$. Then $f \in \mathfrak{M}$, iff

$$\forall a \in \mathbb{R}^+ \; f(a) = 0 \Leftrightarrow a = 0, \quad \text{and}$$

$$\forall a,b,c \in \mathbb{R}^+ \; |a - b| \leq c \leq a + b \Rightarrow f(a) \leq f(b) + f(c) .$$

Corollary 1. (See [1; Lemma 2.5 and Corollary 2.6].) Let $f \in \mathfrak{M}$. Then

$$\forall a, b \in \mathbb{R}^+ \; f(a+b) \leq f(a) + f(b) ,$$

$$\forall a, b \in \mathbb{R}^+ \; a \leq 2b \Rightarrow f(a) \leq 2f(b) ,$$

$$\forall a \in \mathbb{R}^+ \; \forall n \in \mathbb{N} \; 2^{-n} f(a) \leq f(2^{-n}a) ,$$

$$\forall a, b \in \mathbb{R}^+ \; |f(a) - f(b)| \leq f(|a - b|) .$$

Proposition 2. (See [1; Proposition 1.2].) Let a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ have the following properties:

$$\forall a \in \mathbb{R}^+ \ f(a) = 0 \iff a = 0, \quad \text{and}$$

$$f \text{ is concave} .$$

Then f is metric preserving.

Proposition 3. (See [1; Proposition 1.3].) Let a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ have the following properties:

$$f(0) = 0 ,$$

$$\exists a > 0 \quad \forall x > 0 \quad a \leq f(x) \leq 2a .$$

Then f is metric preserving.

Proposition 4. (See [1; Proposition 2.6].) Let $f \in \mathfrak{M}$. Let $d,k > 0$. Let $g(x) = kx$ for $x \in [0,d)$ and $g(x) = f(x)$ for $x \in [d,\infty)$. Then $g \in \mathfrak{M}$ iff

$$f(d) = kd ,$$

$$\forall x, y \in [d,\infty) \ |f(x) - f(y)| \leq k \, |x - y| .$$

Proposition 5. (See [1; Corollary 2.22].) Let $f_n \in \mathfrak{M}$ $(n = 1,2,\ldots)$. If $\sum_{n=1}^{\infty} f_n$ converges to f, then $f \in \mathfrak{M}$.

Proposition 6. (See [1; Proposition 2.23].) Let $\emptyset \neq \mathcal{L} \subset \mathfrak{M}$. Suppose $\mathcal{L}_x = \{f(x) : f \in \mathcal{L}\}$ is a bounded set for every positive x. Let $g(x) = \sup \mathcal{L}_x$ for each $x \in \mathbb{R}^+$. Then g is metric preserving.

Proposition 7. (See [1; Theorem 2.9].) Let $f \in \mathfrak{M}$. Then the following three conditions are equivalent:

$$f \text{ is continuous} ,$$

$$f \text{ is continuous at } 0 ,$$

$$\forall \varepsilon > 0 \quad \exists x > 0 \quad f(x) < \varepsilon .$$

Proposition 8. (See [2; Corollary 3].) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be differentiable on some neighborhood of ∞. If $\lim_{x\to\infty} f'(x) = \infty$, then f is not metric preserving.

The following example shows that the assumption "$\lim_{x\to\infty} f'(x) = \infty$" in Proposition 8 cannot be replaced by "$\limsup_{x\to\infty} f'(x) = \infty$".

Example 1. There is a function $f \in \mathfrak{M}$ such that

(1) f is continuous ,

(2) $f'(0) = \infty$,

(3) f is differentiable on $(0,\infty)$,

(4) $\limsup_{x\to\infty} f'(x) = \infty$.

Put $a_i = 1 - \sqrt{1 - 2^{-2i}}$ $(i = 1,2,3,\ldots)$. For $i = 1,2,3,\ldots$ let

$$h_i(x) = \begin{cases} 0 & \text{for } x = 0 , \\ 2^{-i-1} & \text{for } x \in (0,a_{i+1}) \\ 2^{-i-2}\left[3 + \sin\left(\frac{\pi}{2}\,\frac{2x - a_j - a_{j+1}}{a_i - a_{i+1}}\right)\right] & \text{for } x \in [a_{i+1}, a_i) , \\ 2^{-i} & \text{for } x \in [a_i,\infty) . \end{cases}$$

Since $2^{-i-1} \leq h(x) \leq 2(2^{-i-1})$ for each $x > 0$, by Proposition 3 $h_i \in \mathfrak{M}$. For $i = 1,2,3,\ldots$ let

$$g_i(x) = \begin{cases} (2^{i+1} a_{i+1})^{-1} x & \text{for any } x \in [0,a_{i+1}) , \\ h_i(x) & \text{for } x \in [a_{i+1},\infty) . \end{cases}$$

Since $|h_i'(x)| \leq 2^{-i-2}\pi(a_i - a_{i+1})^{-1} \leq (2^{i+1} a_{i+1})^{-1}$ for each $x \in [a_{i+1},\infty)$, by Proposition 4 $g_i \in \mathfrak{M}$. For $i = 1,2,3,\ldots$ let

$$r_i(x) = \begin{cases} 0 & \text{for} \quad x = 0\ , \\ 2^{-i-1}\ [3 + \cos \frac{2(x-i-1)}{a_i}] & \text{for} \quad x \in [i + 1 - \frac{\pi}{4} a_i\ ,\ i + 1 + \frac{\pi}{4} a_i] \\ 2^{-i} & \text{otherwise}\ . \end{cases}$$

Since $2^{-i} \leq r_i(x) \leq 2(2^{-i})$ for each $x > 0$, by Proposition 3 $r_i \in \mathfrak{M}$. For $i = 1,2,3,\ldots$ let

$$s_i(x) = \begin{cases} (2^i a_i)^{-1} x & \text{for} \quad x \in [0,a_i)\ , \\ r_i(x) & \text{for} \quad x \in [a_i,\infty)\ . \end{cases}$$

Since $|r'_i(x)| \leq (2^i a_i)^{-1}$ for each $x > 0$, by Proposition 4 $s_i \in \mathfrak{M}$. For $n = 1,2,3,\ldots$ let

$$t_n(x) = \sup_{i \geq n} g_i(x) \quad \text{for each} \quad x \in \mathbb{R}^+\ .$$

By Proposition 6 we get $t_n \in \mathfrak{M}$. Now let

$$f_0(x) = \begin{cases} \sqrt{2x - x^2} & \text{for} \quad x \in \langle 0,1)\ , \\ 1 & \text{for} \quad x \in [1,\infty)\ . \end{cases}$$

By Proposition 2 we have $f_o \in \mathfrak{M}$. Further for $n = 1,2,3,\ldots$ let $f_n(x) = \max\{t_n(x), s_n(x)\}$ for each $x \in \mathbb{R}^+$. By Proposition 6 $f_n \in \mathfrak{M}$. Finally let $f(x) = \sum_{n=0}^{\infty} f_n(x)$ for each $x \in \mathbb{R}^+$. By Proposition 5 we get $f \in \mathfrak{M}$. By a routine calculation we can verify that (1) - (4) hold.

<u>Proposition 9</u>. (See [2; Proposition 8].) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be differentiable and let f' be continuous at 0. If f is metric preserving, then it is increasing on some neighborhood of 0.

The following example shows that the assumption "f' is continuous at the point 0" in Proposition 9 is essential.

Example 2. There is a function $f \in \mathfrak{M}$ such that

(5) f is differentiable

(6) f' is continuous on $(0,\infty)$,

(7) f is not increasing on any neighborhood of 0 ,

(8) each neighborhood of 0 contains an interval on which f is strictly convex.

For $n = 1,2,3,\ldots$ put $r_n = \dfrac{(2^n-1)(n+1)}{(2n+1)n^2}$ and let

$$g_n(x) = \begin{cases} 0 & \text{for } x = 0 , \\ a_n x^3 + b_n x^2 + c_n x + d_n & \text{for } x \in [r_n, n^{-1}) , \\ (2-n^{-1})\, n^{-1} & \text{otherwise} , \end{cases}$$

where

$$a_n = (16n^7 + 24n^6 + 8n^5 - 2n^4 - n^3)(n + 1)^{-1}$$

$$b_n = (-48n^6 - 72n^5 - 12n^4 + 18n^3 + 2n^2 - 2n)(n + 1)^{-1} ,$$

$$c_n = (48n^6 + 72n^5 - 30n^3 + n^2 + 5n - 1)(n^2 + n)^{-1} ,$$

$$d_n = (-16n^4 - 8n^3 + 12n^2 + 2n - 2)n^{-1} .$$

By Proposition 3 $g_n \in \mathfrak{M}$. Further for $n = 1,2,3,\ldots$ let

$$f_n(x) = \begin{cases} (2-n^{-1})x & \text{for } x \in [0,r_n) , \\ g_n(x) & \text{for } x \in [r_n,\infty) . \end{cases}$$

By Proposition 4 $f_n \in \mathfrak{M}$. Let $f_0(x) = x$ for each $x \in \mathbb{R}^+$. Finally, let $f(x) = \sup\{f_n(x);\ n = 0,1,2,\ldots\}$ for each $x \in \mathbb{R}^+$.

By Proposition 6 $f \in \mathfrak{M}$. By a routine calculation we can verify that (5) - (8) hold.

Proposition 10. (See [2; Proposition 5].) If a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous at 0, if $f(0) = 0$ and if f is strictly convex at 0, then f is not metric preserving.

Proposition 11. (See [2; Proposition 6].) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be twice differentiable on $\mathbb{R}^+$ and satisfy the following properties:

$f(x) = 0 \Longleftrightarrow x = 0$,

$f'(x) \geqslant 0$ for all $x \geqslant 0$,

there is number $h > 0$ such that $f''(x) \geqslant 0$ for all $x \in [0,h]$ and there is an $x_0 \in [0,h]$ such that $f''(x_0) > 0$.

Then f is not metric preserving.

We shall generalize these assertions.

Theorem 1. Let $f \in \mathfrak{M}$ and $h > 0$. If f is convex on $[0,h]$, then f is linear on $[0,h]$.

Proof. From the convexity we obtain

$$\forall a, b \in \mathbb{R}^+ \ 0 < a \leqslant b \leqslant h \Rightarrow \frac{f(a)}{a} \leqslant \frac{f(b)}{b} . \tag{9}$$

We shall show that $f(x) = \frac{f(h)}{h}x$ for each $x \in [0,h]$. Let $x \in (0,h]$. Let n be a positive integer such that $2^{-n} h \leqslant x$. Then according to (9) and Corollary 1 $f(2^{-n} h) = 2^{-n} f(h)$. Then $\frac{f(h)}{h} = \frac{f(2^{-n}h)}{2^{-n}h} \leqslant \frac{f(x)}{x} \leqslant \frac{f(h)}{h}$, which yields $f(x) = \frac{f(h)}{h} x$.

Now we shall show that metric preserving functions have a derivative at 0.

Lemma 1. Let $f \in \mathfrak{M}$. Suppose there are $h, k > 0$ such that $f(x) \leqslant kx$ for each $x \in [0,h]$. Then

$$\forall x \in \mathbb{R}^+ \ f(x) \leqslant kx \quad \text{and} \tag{10}$$

$$\forall x, y \in \mathbb{R}^+ \ |f(x) - f(y)| \leqslant k \, |x - y| . \tag{11}$$

Proof. Let $x \in \mathbb{R}^+$. Let n be a positive integer such that $2^{-n}x \leq h$. By Corollary 1 $2^{-n} f(x) \leq f(2^{-n} x) \leq k\, 2^{-n} x$, which yields (10). Observe that (11) follows from Corollary 1 and (10).

Lemma 2. Let $f \in \mathfrak{M}$, $k > 0$. If in every neighborhood of 0 there is a point a such that $f(a) = ka$, then $f(x) = kx$ holds in a suitable neighborhood of 0.

Proof. Let $h > 0$ be such that $f(h) = kh$. We shall show that $f(x) = kx$ for each $x \in [0,h]$. Assume that $f(x) \neq kx$ for some $x \in (0,h)$. We distinguish two cases.

1.) Suppose that $f(x) > kx$. Put $A = \{y \in \mathbb{R}^+ : f(y) = ky\}$. Since f is continuous (by Proposition 7), the set $A \cap [0,x]$ is closed and bounded. Hence $m = \max(A \cap [0,x]) \in \mathbb{R}$. Let $y \in A$ be such that $0 < y < x - m$. Then by Corollary 1 $f(m+y) \leq f(m) + f(y) = km + ky = k(m+y)$. Since $f(x) > kx$ and since f is continuous, there is $z \in [m+y, x]$ such that $f(z) = kz$, which contradicts the definition of m.

2.) Suppose that $f(x) < kx$. Since the set $A \cap [x,h]$ is closed and bounded, $M = \min(A \cap [x,h]) \in \mathbb{R}$. Let $r \in A$ be such that $0 < r < M - x$. Then by Corollary 1 $kM = f(M) \leq f(M-r) + f(r) = f(M-r) + kr$, which yields $f(M-r) \geq kM - kr = k(M-r)$. Since $f(x) < kx$ and f is continuous, there is $s \in [x, M-r]$ such that $f(x) = ks$, which contradicts the definition of M.

Lemma 3. Let $f \in \mathfrak{M}$. Then

$$\forall x,\ y > 0 \quad x \geq y \Rightarrow \frac{f(x)}{x} \leq 2\,\frac{f(y)}{y}\ .$$

Proof. Let $x \geq y > 0$. Then $xy^{-1} \geq 1$. Let n be a positive integer such that $2^{n-1} \leq xy^{-1} < 2^n$. Then $2^{1-n} x < 2y$. Therefore by Corollary 1 $f(2^{1-n} x) \leq 2f(y)$. By Corollary 1 $2^{1-n} f(x) \leq f(2^{1-n} x) \leq 2f(y)$. Thus $f(x) \leq 2^{n-1}\, 2f(y) \leq xy^{-1}\, 2f(y)$. From this we get $\frac{f(x)}{x} \leq 2\,\frac{f(y)}{y}$.

Theorem 2. Let $f \in \mathfrak{M}$. Then $f'(0)$ exists (finite or infinite) and

$$f'(0) = \inf\{k > 0 : f(x) \leq kx \text{ for each } x \in \mathbb{R}^+\}\ .$$

Proof. Put $K_f = \{k > 0 : f(x) \leq kx$ for each $x \in \mathbb{R}^+\}$. We distinguish two cases.

1.) Suppose that $K_f \neq \emptyset$. By Proposition 7 the function f is continuous. Hence K_f is closed. Put $k_0 = \inf K_f$. Then $k_0 \in K_f$ and $k_0 > 0$. We shall show that

$$(12) \qquad k_o = \lim_{x \to 0} \frac{f(x)}{x} .$$

Let $\varepsilon > 0$. Then

$$(13) \qquad \forall h > 0 \quad \exists x \in [0,h] \ f(x) > (k_0 - \varepsilon)x .$$

Indeed if not, we have $k_0 - \varepsilon \in K_f$, which contradicts the definition of k_0. We shall show that

$$(14) \qquad \exists h > 0 \quad \forall x \in (0,h] \ f(x) > (k_0 - \varepsilon)x .$$

Suppose that

$$(15) \qquad \forall h > 0 \quad \exists x \in (0,h] \ f(x) \leq (k_0 - \varepsilon)x .$$

Let $h > 0$. Then by (13) there is $x_1 \in (0,h]$ such that $f(x_1) > (k_0 - \varepsilon)x_1$ and by (15) there is $x_2 \in (0,h]$ such that $f(x_2) \leq (k_0 - \varepsilon)x_2$. By the continuity of f there is $x_3 \in (0,h]$ such that $f(x_3) = (k_0 - \varepsilon)x_3$. By Lemma 2 $f(x) = (k_0 - \varepsilon)x$ holds on some neighborhood of 0, which contradicts (13). Since $k_0 \in K_f$, we have $f(x) < (k_0 + \varepsilon)x$ for each $x > 0$. Thus by (14) we obtain $\forall \varepsilon > 0 \ \exists h > 0 \ \forall x \in (0,h] \ k_0 - \varepsilon < \frac{f(x)}{x} < k_0 + \varepsilon$, i.e. (12) holds.

2.) Suppose that $K_f = \emptyset$ (which yields $\inf K_f = \infty$). Let $n \in \mathbb{N}$. Let $h > 0$ be such that $f(h) > 2nh$. Let $x \in (0,h]$. By Lemma 3 $\frac{f(x)}{x} \geq \frac{1}{2} \frac{f(h)}{h} \geq \frac{1}{2} 2n = n$. Therefore $\forall n \in \mathbb{N} \ \exists h > 0 \ \forall x \in (0,h] \ \frac{f(x)}{x} \geq n$, i.e. $f'(0) = \infty$.

Theorem 3. Let $f \in \mathfrak{M}$. Let $f'(0) < \infty$. Then

$$(16) \qquad \forall x \in \mathbb{R}^+ \ f(x) \leq f'(0)x , \quad \text{and}$$

$$(17) \qquad \forall x, y \in \mathbb{R}^+ \ |f(x) - f(y)| \leq f'(0)\,|x - y| .$$

Proof. Let $\varepsilon > 0$. Then there is $h > 0$ such that $f(x) \leq [f'(0) + \varepsilon]x$ for each $x \in [0,h]$. By Lemma 1 $f(x) \leq [f'(0) + \varepsilon]x$ for each $x \in \mathbb{R}^+$. Since $\varepsilon > 0$ was arbitrary, (16) holds.

Observe that (17) follows from Corollary 1 and (16).

Corollary 2. (Compare [2; Lemma 5].) Let $f \in \mathfrak{M}$ be differentiable. Then $|f'(x)| \leq f'(0)$ for each $x \in \mathbb{R}^+$.

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