INROADS

Real Analysis Exchange Vol. 13 (1987-88)

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Some Answers to a Question of P. Bullen

A question posed by Peter Bullen [1] is whether it is possible to restrict the guage function used in generalized Riemann type integrals. Here we investigate the guage function needed for the Perron integral (equivalent to the Riemann-Complete and narrow sense Denjoy integral), the Lebesgue-Stieljes integral and the Lebesgue integral for bounded measurable functions. The Henstock or Riemann-Complete (R-C) integral integrates a function f assumed to be finite valued. The R-C integral of f on $[a,b]$ is L provided that for each $\varepsilon > 0$ there is a positive function δ such that $|\sum f(z_i)\Delta x_i - L| < \varepsilon$ whenever $a = x_0 < x_1 < \dots < x_n = b$ is a partition of $[a,b]$, $z_i \in [x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1} < \delta(z_i)$. The function δ is called the guage function.

Henstock [2, p.127] showed how to determine δ for the Perron integral. Utilizing the majorant and minorant, he found δ and showed that the Perron integral is contained in the R-C integral. However, the character of δ has not been determined.

Here we utilize the equivalence of these integrals with the narrow sense Denjoy Integral, the fact that

$$F(x) = D^*\int_a^x f(t)dt \quad \text{is} \quad ACG^*$$

and that $F'(x) = f(x)$ a.e. (cf.[5]) to determine the character of the δ needed for the Perron Integral. Given f, a D^* integrable function on $[a,b]$ and $F(x) = \int_a^x f(t)dt$, let Z_f be the set of x where $F'(x)$ does not exist or $F'(x) \neq f(x)$. Let Z be a G_δ subset of measure 0 containing Z_f. We then have the following results concerning the guage function δ needed to integrate f.

Theorem 1. The guage function for an R-C integrable function f can be chosen to be measurable and δ restricted to the complement of Z can be chosen to be Baire 2.

Example 1. There is a Lebesgue integrable function f for which no Baire function δ will suffice to estimate the integral.

Theorem 2. If F ACG^* and $f(x) = F'(x)$ wherever $F'(x)$ exists and $f(x) = 0$ otherwise, a Baire 2 δ will suffice to estimate the integral of f. Alternatively, if $|f|$ is dominated by a Baire function, a Baire δ will suffice.

Theorem 3. *If* f *is a bounded measurable function on* $[a,b]$ *the guage needed to approximate* $\int_a^b f(x)dx$ *need only be chosen from Baire class* 2.

We proceed with the proof of these theorems. We will then continue with an investigation of the Lebesgue-Stieltjes integral determined by the $D^{\#}$ integration basis and a guage function.

Proof of Theorem 1. Suppose f is D^* integrable on $[a,b]$ and $F(x) = \int_a^x f(t)dt$. Since F is ACG^* there exists a sequence of closed sets E_k such that $[a,b] = \cup E_k$ and F is AC^* on each E_k. Let $\{I_{kj}\}$ be the set of intervals contiguous to E_k. Since F is AC^* on E_k, given $\varepsilon > 0$ there is a natural number N_k such that

$$(*) \qquad \sum_{j=N_k}^{\infty} \mathcal{O}(F;I_{kj}) < \frac{\varepsilon}{2^k} .$$

(Here $\mathcal{O}(F;I)$ is the oscillation of F on I.) Letting $\mathring{I}$ denote the interior of I, we have that each

$$A_k = E_k \cup \bigcup_{j=N_k}^{\infty} I_{kj} = [a,b] \setminus \bigcup_{j=1}^{N_k-1} \mathring{I}_{kj}$$

is a finite union of closed intervals. Since F is AC^* on E_k, there is $\delta_k > 0$ such that if $\sum |I_m| < \delta_k$ where the I_m are nonoverlapping intervals with endpoints in E_i, then

$$\sum|F(I_m)| < \varepsilon/2^k.$$

Because of (*), whenever $\sum|I_m| < \delta_k$ where I_m are nonoverlapping and each has an endpoint in E_k and each $I_m \subset A_k$, $\sum|F(I_m)| < 3\varepsilon/2^k$. By the continuity of F and the fact that A_k is a finite union of intervals, it is possible to determine $G_k \supset A_k$, G_k open, such that whenever $\{I_m\}$ are a set of nonoverlapping intervals and each I_m contains a point of E_k and $I_m \subset G_k$ with $\sum|I_m| < \delta_k$, we have $\sum|F(I_m)| < 3\varepsilon/2^k$.

Recall that Z_f is the set of x where $F'(x)$ does not exist or $F'(x) \neq f(x)$. Let Z be a G_δ set of measure 0 containing Z_f,

$$Z_o = \{x \in Z: f(x) = 0\}$$

$$Z_n = \{x \in Z: n - 1 < |f(x)| \leq n\}.$$

For positive integers n and natural numbers k, let G_{nk} be open sets with $Z_n \cap E_k \subset G_{nk} \subset G_k$ and with $|G_{nk}| < \varepsilon/(n\ 2^{n+k})$ and $|G_{nk}| < \delta_k$. We now define the guage function δ.

$$\text{If } x \in Z_n \cap E_k \setminus \bigcup_{1}^{k-1} E_i, \text{ let } \delta(x) = \text{dist}(x, G_{nk}^c);$$

$$\text{if } x \notin Z, \text{ let } \delta(x) = \sup\{\delta: \left|\frac{F(I)}{|I|} - f(x)\right| \leq \varepsilon \text{ when } |I| < \delta\}.$$

Note that since Z_n cannot be chosen to be a Borel set, δ is not in general a Baire function.

Consider any acceptable partition for δ; that is, let $a = x_o < x_1 < \dots < x_n = b$ and $z_i \in [x_{i-1}, x_i]$ where $\delta > x_i - x_{i-1}$. Then

$$F(b) - F(a) - \sum f(z_i)\Delta x_i = \sum F(x_i) - F(x_{i-1}) - f(z_i)\Delta x_i.$$

Let $\sum_{nk}$ be summation over all i where $z_i \in Z_n \cap E_k$. Then

$$\sum_{nk}|f(z_i)\Delta x_i| \leq \sum n\Delta x_i < \varepsilon/2^{n+k}$$

where $\sum$ is over the $[x_{i-1}, x_i] \subset G_{nk}$. Letting $\sum_n$ be the summation over all i where $z_i \in Z_n$, we have

$$\sum_n|f(z_i)\Delta x_i| < 2\varepsilon/2^n.$$

If $\sum^1$ is the summation over all i where $z_i \in \cup Z_n$, we have $\sum^1|f(z_i)\Delta x_i| < 4\varepsilon$. Also $\sum^1|F(x_i) - F(x_{i-1})| < \sum_k 3\varepsilon/2^k = 6\varepsilon$. Letting $\sum^2$ be the summation over all i where $z_i \notin Z$,

$$|\sum^2 F(x_i) - F(x_{i-1}) - f(z_i)\Delta x_i| \leq \sum^2 \varepsilon x_i = \varepsilon(b - a)$$

Thus

$$\begin{aligned}|\sum F(x_i) - F(x_{i-1}) - f(z_i)\Delta x_i| &\leq \sum^1|F(x_i) - F(x_{i-1})| + \sum^1|f(z_i)\Delta x_i| \\ &\quad + |\sum^2 F(x_i) - F(x_{i-1}) - f(z_i)\Delta x_i| \\ &\leq 10\varepsilon + \varepsilon(b - a).\end{aligned}$$

It follows that δ is an appropriate guage for estimating the integral of f. To complete the proof of Theorem 1, the nature of δ must be examined. Actually we will consider a guage smaller than δ. Given $\varepsilon > 0$, for $x \in Z^c$ let

$$N(x) = \min\{N: |\frac{F(I)}{|I|} - f(x)| \le \varepsilon \text{ when } x \in I = [\frac{p}{q}, \frac{r}{s}] \text{ and } |I| < \frac{1}{N}\}$$

for $x \in Z$ let $N(x) = 0$.

Let

$$\delta_o(x) = \begin{cases} \delta(x) & \text{if } x \in Z \\ \frac{1}{N(x)} & \text{if } x \in Z^c \end{cases} .$$

Note that $\delta_o \le \delta$. Then $Z = N^{-1}(\{0\})$ is a G_δ. Furthermore $N^{-1}([1,m]) =$

$$\{x: \forall p,q,r,s, \ x \in (\frac{p}{q}, \frac{r}{s}) = I \text{ and } |I| < \frac{1}{m} \Rightarrow |\frac{F(I)}{|I|} - f(x)| < \varepsilon\} \setminus Z$$

$$= \bigcap \{x: x \in (\frac{p}{q}, \frac{r}{s}) = I \Rightarrow x \in f^{-1}([\frac{F(I)}{|I|} - \varepsilon, \frac{F(I)}{|I|} + \varepsilon])\} \setminus Z$$

$$= \bigcap \{x: x \in (\frac{p}{q}, \frac{r}{s}) = I \ \text{ or } \ x \in f^{-1}([\frac{F(I)}{|I|} - \varepsilon, \frac{F(I)}{|I|} + \varepsilon])\} \setminus Z$$

where the last intersections are over all p, q, r, s, with

$$|\frac{p}{q} - \frac{r}{s}| < \frac{1}{m} .$$

Since f is Baire 1 on Z^c, the intersection is a G_δ subset of Z^c. Thus $N^{-1}(\{m\})$ is a $G_{\delta\sigma}$ subset of Z^c and thus is a $G_{\delta\sigma}$ set. Thus N is in Baire class 2 because $N^{-1}(G)$ is a $G_{\delta\sigma}$ for each open set G. Let $d_1(x) = 1/N(x)$, $x \in Z^c$; $d_1(x) = 0$, otherwise. Let $d_2(x) = \delta(x)$, $x \in Z$; $d_2(x) = 0$, otherwise. Then d_2 is measurable since it is defined on a set of measure 0 and d_1 is Baire 2. Thus

$$\delta = d_1 + d_2$$

can be chosen measurable.

The proof of Theorem 2 follows easily from the above. We only need note that if $|f|$ is dominated by a Baire function b, the sets Z_n in the proof can be replaced by

$$Z'_n = \{x \in Z: n-1 < |b(x)| \leq n\}$$

and the resulting δ is a Baire function (of Baire class 2 or the same class as b if b is Baire α with $\alpha > 2$). The proof of Theorem 2 where $f(x) = 0$ when $F'(x)$ does not exist and the proof of Theorem 3 are obtained by letting $\delta = \varepsilon$ on Z. This produces a Baire 2 guage δ_o. However, the set of measure 0 is crucial in determining that a Baire guage can be used. The following construction for Example 1 shows that there is a function defined on a set of measure 0 whose integral (which is 0) cannot be estimated by a Baire guage. Let C be the Cantor ternary set and let $\{B_\alpha\}_{\alpha<\omega_c}$ be a well ordering of the uncountable Borel subsets of C. Let $x_1^o, x_2^o, \ldots, x_n^o, \ldots$ be a countable subset of B_o. In general if $x_1^\beta, x_2^\beta, \ldots, x_n^\beta, \ldots$ is a countable subset of B_β and all x_i^β are distinct for i, β with $\beta < \alpha$, then it is possible to choose distinct

$$x_i^\alpha \in B_\alpha \setminus \bigcup_{\beta<\alpha} \bigcup_{i=1}^{\infty} \{x_i^\beta\}$$

because B_α has cardinality c and $\bigcup_{\beta<\alpha} \bigcup_{i=1}^{\infty} \{x_i^\beta\}$ has cardinality less than c. Define

$$f(x) = \begin{cases} n & \text{if there is an } \alpha \text{ so } x = x_n^\alpha \\ 0 & \text{otherwise} \end{cases}$$

Suppose δ is a positive Baire function. Consider the Baire function $g = 1/\delta$. If g is a Baire function, $g^{-1}((0,N))$ is a Borel set. Since $g^{-1}((0,\infty))$ contains an uncountable subset of the Cantor set, we can choose N so that $g^{-1}((0,N))$ contains an uncountable Borel subset of the Cantor set. By the construction of f there is $x_o \in g^{-1}((0,N))$ so $f(x_o) = N > g(x_o)$. Let I_o be an interval containing x_o of length $\delta(x_o)$ and consider a partition containing I_o. For such a partition,

$$\sum f(z_i)\Delta x_i \geq f(x_o)\delta(x_o) > 1.$$

Since $\int_o^1 f(x)\,dx = 0$ and since there are partitions compatible with δ containing I_o, it follows that the integral of f cannot be approximated with a Baire guage δ even if f is Lebesgue integrable.

We now consider the guage needed for the $D^\#$ derivation basis. This basis gives rise to the Lebesgue-Stieltjes integral (cf.[4]). For a positive function δ,

$\beta_\delta^{\#} = \{(I,x): I$ is an interval and $I \subset (x - \delta(x), x + \delta(x))\}$. The $D^{\#}$ basis is the set of all $\beta_\delta^{\#}$ for various $\delta > 0$. Let $D_B^{\#}$ [$D_M^{\#}$] be the $D^{\#}$ basis with the δ restricted to be Baire [measurable with respect to the Lebesgue-Stieltjes measure m_g] functions. Let g be a monotone non-decreasing function. We prove the following theorems:

Theorem 4. *If f is a bounded Baire [measurable with respect to m_g] function, then*

$$D^{\#}\int f\, dg = D_B^{\#}\int f\, dg .$$

Theorem 5. *If f is a Baire [measurable with respect to m_g] function and f is Lebesgue-Stieltjes integrable with respect to g, then*

$$D^{\#}\int f\, dg = D_B^{\#}\int f\, dg \qquad [= D_M^{\#}\int f\, dg] .$$

To prove Theorem 4 we will utilize a dominated and monotone convergence theorem as given by McShane [3]. When we need them, McShane's theorems will be restated for the integrals under consideration because they were originally stated in an abstract setting.

Proof of Theorem 4. First suppose f is continuous. Then f is Riemann-Stieltjes integrable and a constant function can be used for δ. Suppose the theorem for bounded Baire functions is true for Baire class β when $\beta < \alpha$ and that f is bounded and in Baire class α. Then $f = \lim f_n$, $f_n \in \bigcup_{\beta<\alpha} B_\beta$ and given $\varepsilon > 0$ there are Baire functions δ_n for approximating $\int_a^b f_n \, dg$ within ε. Let M be a bound on f and let

$$E_n = \{x: |f_m - f| < \varepsilon \cdot M \quad \text{whenever} \quad m > n\} .$$

The E_n are an increasing sequence of sets and $\cup E_n = [a,b]$. Let $A_{n,k} = \{x: \delta_k(x) \geq 1/n\} \cap E_k$ and $A_n = \bigcup_k A_{n,k}$, $A_o = \emptyset$. Let $B_n = A_n \setminus A_{n-1}$ and let $\delta(x) = 1/n$ for $x \in B_n$. The proof that f is $D_B^{\#}$ integrable requires the dominated convergence theorem which we now state:

<u>Dominated Convergence Theorem</u>. Assume $\{f_n\}$ are $D_B^{\#}$ integrable with respect to a monotone nondecreasing function g. Assume

$$|f_i - f_j|$$

is $D_B^{\#}$ integrable for each $i, j = 1, 2, \cdots$. Assume there exists an h which is $D_B^{\#}$ integrable with respect to g and $|f_n(x)| \leq h(x)$ for all x and n. Suppose $f_n(x) \to f(x)$ for all x. Then, if for each positive integer j, if for each $\varepsilon > 0$, and if for each sequence $\{\delta_n\}$ in $D_B^{\#}$, there exists a δ in $D_B^{\#}$ such that for each $(x,I) \in \beta_\delta^{\#}$ there corresponds a positive integer $j(x,I) \geq j$ such that $(x,I) \in \beta^{\#}_{\delta_{j(x,I)}}$ and $|f_i(x) - f(x)| < \varepsilon \cdot h(x)$ for all $i \geq j(x,I)$; then, f is $D_B^{\#}$

integrable and

$$\lim_{n\to\infty} D_B^{\#}\int f_n \, dg \;=\; D_B^{\#}\int f \, dg \; .$$

We return to the proof of Theorem 4.

If $(x,I) \in \beta_{\delta}^{\#}$, there exists an n such that $x \in B_n$ and $\delta(x) = 1/n$. Therefore $x \in A_n$ which implies there exists a k such that $x \in A_{n,k}$. Therefore, $\delta_k(x) \geq 1/n$ and $x \in E_k$, so $(x,I) \in \beta_{\delta_k}^{\#}$ and

$$|f_m(x) - f(x)| < \varepsilon \cdot M$$

for all $m \geq k$. The theorem is thus true for all Baire classes α.

We note that if f is bounded and measurable with respect to m_g, there is a Baire 2 function $\tilde{f}$ which is bounded and equal to f on the complement of a G_δ set of m_g measure 0. The proof for $D_M^{\#}$ follows by letting $\delta(x) = \delta$ on an open set G containing this G_δ set and having small m_g measure.

<u>Proof of Theorem</u> 5. Without loss of generality suppose f is a nonnegative Baire function [measurable m_g] and Lebesgue-Stieltjes integrable, and let $f = \lim f_n$ where $f_n = \min(n, f(x))$. We use the sets $E_n = \{x: f(x) < n\}$, and A_{nk}, A_n, and B_n as in the proof of Theorem 4. Here the corresponding monotone convergence theorem of McShane is required.

Monotone Convergence Theorem. Let g be a monotone nondecreasing function and $0 \leq f_1(x) \leq f_2(x) \leq \cdots$ such that

$$\lim f_n(x) = f(x) < +\infty$$

for each x. Assume $\{f_n\}$ are $D_B^{\#}$ $[D_M^{\#}]$ integrable with respect to g. Then, if for each $0 < \varepsilon < 1$ and if for any sequence $\{\delta_n\}$ in $D_B^{\#}$ $[D_M^{\#}]$ there exists a δ in $D_B^{\#}$ $[D_M^{\#}]$ such that if $(x,I) \in \beta_\delta^{\#}$ there exists an $n(x,I)$ for which

$$(x,I) \in \beta^{\#}_{\delta_{n(x,I)}}$$

and $f_{n(x,I)}(x) \geq \varepsilon \cdot f(x)$, then f is $D_B^{\#}$ $[D_M^{\#}]$ integrable and

$$\lim_{n\to\infty} D_B^{\#}\int f_n \, dg = D_B^{\#}\int f \, dg \qquad [\lim_{n\to\infty} D_M^{\#}\int f_n \, dg = D_M^{\#} \int f \, dg] .$$

We return to the proof of Theorem 5. If $(x,I) \in \beta_\delta^{\#}$, then there exists an n such that $\delta(x) = 1/n$ and $x \in B_n$. Therefore $x \in A_n$ which implies there exists a k such that $x \in A_{n,k}$. Therefore $\delta_k(x) \geq 1/n$ and $f(x) < k$. Hence $f_k(x) = f(x) > \varepsilon \cdot f(x)$.

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Received April 10, 1987