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Małgorzata Filipczak, Institute of Mathematics, University of Łódź, Łódź, Poland

ON GENERATORS FOR BOREL SETS

Let $\mathcal{A}$ be a collection of subsets of X. The smallest σ-algebra on X containing $\mathcal{A}$ will be denoted by $\sigma(\mathcal{A})$; $\mathcal{A}$ is called a generating family (or generator) for $\mathcal{B}$ if $\sigma(\mathcal{A}) = \mathcal{B}$. $\mathcal{A}$ is said to separate two points $x,y \in X$ if there is a set $G \in \mathcal{A}$ which contains one of them but not the other. $\mathcal{A}$ is said to be a separating family if it separates any two distinct points of X.

The terminology and definitions concerning topology come from the book, "General Topology", by R. Engelking [1]. A topological space X is called:

- a locally compact space if for each $x \in X$ there exists a neighborhood U of x such that $\bar{U}$ is a compact subspace of X,
- a perfectly normal space if X is a normal space and each closed subset of X is a G_γ-set.

Compact (and σ-compact) spaces are assumed to be T_2. A set in a linear space E is convex if, whenever it contains points x and y, it also contains the line segment joining x and y, i.e. the set $\{tx + (1-t)y : t \in [0,1]\}$. If X is a topological space, then the natural Borel structure on X (generated by the family of all open subsets of X) will be denoted by $\mathcal{B}_X$. ω stands for the first infinite ordinal; Ω, for the first uncountable ordinal.

K.P.S. Bhaskara Rao and B.V. Rao in [4, p. 19] have stated:

> "The family $\mathcal{J}$ of all open intervals of $\mathbb{R}$ is a generator for $\mathcal{B}_{\mathbb{R}}$. A subfamily $\mathcal{J}_0 \subset \mathcal{J}$ is a generator for $\mathcal{B}_{\mathbb{R}}$ iff the set of end points of intervals in $\mathcal{J}_0$ is dense in $\mathbb{R}$. Thus if $\mathcal{J}_0 \subset \mathcal{J}$ is a generator for $\mathcal{B}_{\mathbb{R}}$, then, by removing any finite set of $\mathcal{D}$ intervals from $\mathcal{J}_0$, we still get a generator for $\mathcal{B}_{\mathbb{R}}$".

However, this statement is false, as is easily seen from examples 1 and 2 below.

In the first place we shall prove:

LEMMA 1. Let $\mathcal{B}$ be a σ-algebra of subsets of X. If a family $\mathcal{G} \subset \mathcal{B}$ is a generator for $\mathcal{B}$ and $\mathcal{B}$ separate points $x,y \in X$, then $\mathcal{G}$ also separates these points; that is, there is a $G \in \mathcal{G}$ such that $x \in G$ and $y \notin G$ or $x \notin G$ and $y \in G$.

PROOF: Suppose that $\mathcal{G}$ does not separate x from y. So for each $G \in \mathcal{G}$, $x \in G$ and $y \in G$ or $x \notin G$ and $y \notin G$. The family of all subsets of X satisfying the above condition forms a σ-algebra containing $\sigma(\mathcal{G}) = \mathcal{B}$. So for any $A \in \mathcal{B}$ $x \in A$ and $y \in A$ or $x \notin A$ and $y \notin A$, which ends the proof.

It follows from Lemma 1 that if the family $\mathcal{F}_0 \subset \mathcal{F}$ is a generator for $\mathcal{B}_{\mathbb{R}}$, then the set of end points of intervals in $\mathcal{F}_0$ is dense in $\mathbb{R}$. (The supposition that there exists a nonempty open set U which contains no end point of any interval in $\mathcal{F}_0$ implies that the family $\mathcal{F}_0$ does not separate any pair of points of U). However, the fact that the set of end points of intervals in $\mathcal{F}_0$ is dense in $\mathbb{R}$ does not imply that $\sigma(\mathcal{F}_0) = \mathcal{B}$.

EXAMPLE 1: Let Q^+ be the set of all positive rational numbers. Let $\mathcal{F}_1 = \{(-w,w) : w \in Q^+\}$. The family $\mathcal{F}_1 \subset \mathcal{F}$ is not a generator for $\mathcal{B}_{\mathbb{R}}$, because it does not separate x and -x. The set of end points of intervals in $\mathcal{F}_1$ forms the set of rational numbers (without zero).

EXAMPLE 2: We shall construct a subfamily of $\mathcal{F}$ which is not a generator for $\mathcal{B}_{\mathbb{R}}$ and both the set of left end points of intervals in this family and the set of right end points are dense in $\mathbb{R}$.

Let $\{w_1,w_2,w_3,\ldots\}$ be the sequence of all rational numbers. We construct a family $\mathcal{F}^k$ of neighborhoods of w_k as follows:

$$\mathcal{F}^k = \left\{\left(w_k - \frac{1}{2^{k+j}}\,,\, w_k + \frac{1}{2^{k+j}}\right) : j \in \mathbb{N}\right\}$$

for $k \in \mathbb{N}$. The measure of the set

$$\cup\, \mathcal{F}^k = \bigcup_{j \geq 1} \left(w_k - \frac{1}{2^{k+j}}\,,\, w_k + \frac{1}{2^{k+j}}\right)$$

$$= \left(w_k - \frac{1}{2^{k+1}}\,,\, w_k + \frac{1}{2^{k+1}}\right)$$

is equal to $\frac{1}{2^k}$. Let

$$\mathcal{F}_2 = \bigcup_{k=1}^{\infty} \mathcal{F}^k = \{I \in \mathcal{F} : I \in \mathcal{F}^k \text{ for some } k \in \mathbb{N}\} .$$

The set $\bigcup\mathcal{F}_2$ has measure less than or equal to 1 and no proper subset of the set $\mathbb{R} - \bigcup\mathcal{F}_2$ belongs to $\sigma(\mathcal{F}_2)$. Therefore $\sigma(\mathcal{F}_2) \neq \mathcal{B}_{\mathbb{R}}$. Moreover both left and right end points of intervals in $\mathcal{F}_2$ lie arbitrarily close to any rational number. Hence these sets are dense in $\mathbb{R}$.

If we know only the set of end points of intervals in a family $\mathcal{F}_0 \subset \mathcal{F}$, we are not able to ascertain that $\sigma(\mathcal{F}_0) = \mathcal{B}_{\mathbb{R}}$. There exist two families contained in $\mathcal{F}$ which have the same sets of right end points and left end points of intervals, and one of them is a generator for $\mathcal{B}_{\mathbb{R}}$ but the second is not.

EXAMPLE 3: Let Q^+ be the set of all positive rational numbers

$$\mathcal{F}_3 = \{(-w,w) : w \in Q^+\} \cup \{(0,1), (-1,0)\} ,$$

$$\mathcal{F}_4 = \{(0,w) : w \in Q^+\} \cup \{(-w,0) : w \in Q^+\} .$$

The left end points of intervals in $\mathcal{F}_3$ and $\mathcal{F}_4$ form the set of nonpositive rational numbers; the right end points, the set of nonnegative rational numbers. Moreover $\sigma(\mathcal{F}_3) \neq \mathcal{B}_{\mathbb{R}}$ (see Example 1), but $\sigma(\mathcal{F}_4) = \mathcal{B}_{\mathbb{R}}$.

Next, we shall formulate a necessary and sufficient condition that a subfamily of $\mathcal{F}$ generates a σ-algebra $\mathcal{B}_{\mathbb{R}}$. We shall first prove Theorem 1.

THEOREM 1: Let X be a topological space such that any open or closed subset in this space is σ-compact. If $\mathcal{F}$ is an arbitrary σ-algebra in X and $\mathcal{F}$ admits the property:

(*) for any $x \in X$ and $y \in X$, $x \neq y$, there is $L \in \mathcal{F}$ such that $x \in \operatorname{Int} L$ and $y \notin \bar{L}$,

then $\mathcal{F}$ contains the Borel algebra $\mathcal{B}_X$.

Proof: For any $x \neq y$, $x,y \in X$ there exist disjoint sets $A,B \in \mathcal{F}$ with $x \in \operatorname{Int} A$ and $y \in \operatorname{Int} B$. (From condition (*) it follows that there is $L_x \in \mathcal{F}$ such that $x \in \operatorname{Int} L_x$, $y \notin \bar{L}_x$ and $L_y \in \mathcal{F}$ such that $y \in \operatorname{Int} L_y$, $x \notin \bar{L}_y$. Put $A = L_x$, $B = L_y - L_x$.)

We shall show that for any disjoint, compact sets K_1, K_2 there exist sets $F_1, F_2 \in \mathcal{F}$ such that $K_1 \subset F_1, K_2 \subset F_2, F_1 \cap F_2 = \emptyset$.

Let $y \in K_2$. For all pairs (x,y), $x \in K_1$ there are sets $A(x)$, $B(y,x)$ with $x \in \operatorname{Int} A(x)$, $y \in \operatorname{Int} B(y,x)$ and $A(x) \cap B(y,x) = \emptyset$. The collection $\{\operatorname{Int} A(x) : x \in K_1\}$ is an open cover of the set K_1. Therefore it has a finite subcover $\{\operatorname{Int} A(x_j) : j = 1,\dots,n\}$. The sets

$$A^*(y) = \bigcup_{j=1}^{n} A(x_j) , \quad B(y) = \bigcap_{j=1}^{n} B(y,x_j)$$

are disjoint and belong to $\mathcal{F}$, and

$$y \in \operatorname{Int} B(y) , \quad K_1 \subset \operatorname{Int} A^*(y) .$$

Now let y run through K_2 and select a finite subcover from $\{B(y) : y \in K_2\}$. We can define

$$F_2 = \bigcup_{j=1}^{m} B(y_j) , \quad F_1 = \bigcap_{j=1}^{m} A^*(y_j) .$$

To complete the proof of Theorem 1 we shall show that each open set in X belongs to $\mathcal{F}$. Suppose we are given an open set G

$$G = \bigcup_{n=1}^{\infty} K_n , \quad X \setminus G = \bigcup_{m=1}^{\infty} L_m$$

where K_n, L_m are compact sets. For any pair K_n, L_m there are sets $F_1^{n,m}$, $F_2^{n,m} \in \mathcal{F}$ separating K_n, L_m and

$$K_n \subset \bigcap_{m=1}^{\infty} F_1^{n,m} \subset G .$$

Then $G = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} F_1^{n,m} \in \mathcal{F}$.

COROLLARY: If in theorem 1, $\mathcal{F} \subset \mathcal{B}_X$, then $\mathcal{F} = \mathcal{B}_X$ if and only if $\mathcal{F}$ admits the property (*).

REMARK: If X is a locally compact, σ-compact and perfectly normal topological space, then any open or closed subset in X is σ-compact.

Obviously if a family $\mathcal{L} \subset \mathcal{B}_X$ admits the property

$$(*') \quad \begin{array}{l} \text{for any distinct points } x,y \in X, \text{ there exists } G \in \mathcal{S} \text{ such that} \\ x \in \operatorname{Int} G \text{ and } y \notin \overline{G} \text{ or } x \notin \overline{G} \text{ and } y \in \operatorname{Int} G, \end{array}$$

then $\sigma(\mathcal{S}) = \mathcal{B}_X$. Unfortunately this condition is not necessary. The family of all open intervals in $\mathbb{R}$ having zero as one end point does not satisfy condition (*') (Zero belongs to the closure of any interval in this family.) but it generates $\mathcal{B}_{\mathbb{R}}$ (because it satisfies condition (*)). On the other hand the condition

$$(**) \quad \begin{array}{l} \text{for any } x,y \in X,\ x \neq y, \text{ there exists } G \in \mathcal{S} \text{ such that} \\ x \in G \text{ and } y \notin G \text{ or } x \notin G \text{ and } y \in G \end{array}$$

(without operations of interior and closure) is not sufficient. The family of all one-point sets in $\mathbb{R}$ satisfies (**) and does not generate $\mathcal{B}_{\mathbb{R}}$. However, the following theorem holds for $\mathbb{R}$:

THEOREM 2: Let $\mathcal{J}_0$ be a family of open intervals in $\mathbb{R}$. $\mathcal{J}_0$ is a generator of $\mathcal{B}_{\mathbb{R}}$ if and only if it satisfies the condition:

$$(**) \quad \begin{array}{l} \text{for any distinct points } x,y \in \mathbb{R}, \text{ there exists } I \in \mathcal{J}_0 \\ \text{that } x \in I \text{ and } y \notin I \text{ or } x \notin I \text{ and } y \in I . \end{array}$$

Proof: The essential part of the proof is Lemma 2:

LEMMA 2: If a family $\mathcal{J}_0$ consisting of open intervals satisfies condition (**), then for any disjoint compact intervals A, B there exist disjoint sets F_A, $F_B \in \sigma(\mathcal{J}_0)$ such that $A \subset F_A$ and $B \subset F_B$.

Proof of Lemma 2: Let A, B be as stated. We may clearly assume that A lies to the left of B (i.e. $x < y$ for any $x \in A$, $y \in B$). Let us consider points $a = \sup A$ and $b = \inf B$. Since the family $\mathcal{J}_0$ satisfies (**), there is a set $F \in \mathcal{J}_0$ such that $a \in F$ and $b \notin F$ or $a \notin F$ and $b \in F$. In the first case $F \cap B = \emptyset$; in the second, $F \cap A = \emptyset$ (since F is an interval).

We shall construct two countable families $\mathcal{L}_A$, $\mathcal{L}_B$ of sets belonging to $\sigma(\mathcal{J}_0)$ with

$$A \subset \bigcup \mathcal{L}_A , \quad B \subset \mathcal{L}_B , \quad (\bigcup \mathcal{L}_A) \cap (\bigcup \mathcal{L}_B) = \emptyset .$$

We shall construct two transfinite sequences $\{F^A_\alpha\}_{\alpha<\Omega}$, $\{F^B_\alpha\}_{\alpha<\Omega}$ of disjoint sets belonging to $\sigma(\mathcal{F}_0)$ which have the following properties:

1° $F^A_\alpha \cap B = \emptyset$, $F^B_\alpha \cap A = \emptyset$

2° $A_\alpha = A - \bigcup_{\beta<\alpha} F^A_\beta$ and $B_\alpha = B - \bigcup_{\beta<\alpha} F^B_\beta$ are subintervals in A and B

3° $(A \neq \emptyset$ or $B \neq \emptyset) \iff (F^A_\alpha \neq \emptyset$ or $F^B_\alpha \neq \emptyset)$ for each $0 \leq \alpha < \Omega$.

Let $\alpha = 0$. If $F \cap B = \emptyset$, then we put $F^A_0 = F$ and $F^B_0 = \emptyset$. If $F \cap A = \emptyset$, then $F^A_0 = \emptyset$ and $F^B_0 = F$.

Let α be a fixed ordinal number less than Ω and suppose that we have sets F^A_β, F^B_β with the required properties for each $\beta < \alpha$.

Let us consider the sets $A_\alpha = A \setminus \bigcup_{\beta<\alpha} F^A_\beta$ and $B_\alpha = B - \bigcup_{\beta<\alpha} F^B_\beta$. There are subintervals in A and B (as intersections of intervals).

1. If $A_\alpha = \emptyset$ and $B_\alpha = \emptyset$, then we put $F^A_\alpha = \emptyset$ and $F^B_\alpha = \emptyset$

2a. If $A_\alpha = \emptyset$ and $B_\alpha \neq \emptyset$, then $F^A_\alpha = \emptyset$, $F^B_\alpha = \mathbb{R} - \bigcup_{\beta<\alpha} (F^A_\beta \cup F^B_\beta)$

b. If $A_\alpha \neq \emptyset$ and $B_\alpha = \emptyset$, then $F^A_\alpha = \mathbb{R} - \bigcup_{\beta<\alpha} (F^A_\beta \cup F^B_\beta)$, $F^B_\alpha = \emptyset$

3. If $A_\alpha \neq \emptyset$ and $B_\alpha \neq \emptyset$, let $a_\alpha = \sup A_\alpha$ and $b_\alpha = \sup B_\alpha$. Then $a_\alpha < b_\alpha$.

Since the family $\mathcal{F}_0$ satisfies (**), there is a set $F \in \mathcal{F}_0$ with $a_\alpha \in F$ and $b_\alpha \in F$ or $a_\alpha \notin F$ and $b_\alpha \in F$. Suppose that $a_\alpha \in F$ and $b_\alpha \notin F$. We put $F^B_\alpha = \emptyset$ and $F^A_\alpha = F - \bigcup_{\beta<\alpha} (F^A_\beta \cup F^B_\beta)$. The set F^A_α contains the nonempty interval $F \cap A_\alpha$; so $F^A_\alpha \neq \emptyset$. Moreover $F^A_\alpha \cap B \subset (F^A_\alpha \cap \bigcup_{\beta<\alpha} F^B_\beta) \cup (F^A_\alpha \cap B_\alpha) = F^A_\alpha \cap B_\alpha \subset F \cap B_\alpha = \emptyset$. If $a_\alpha \notin F$ and $b_\alpha \in F$, then we analogously define $F^A_\alpha = \emptyset$ and $F^B_\alpha = F - \bigcup_{\beta<\alpha} (F^A_\beta \cup F^B_\beta)$.

Both of the sequences $\{F^A_\alpha\}_{\alpha<\Omega}$, $\{F^B_\alpha\}_{\alpha<\Omega}$ are, from a certain $\beta < \Omega$, equal to $\emptyset$. Suppose it is not. Then there is an uncountable descended sequence of subintervals A_α (B_α) in the interval A (or B), which gives a contradiction.

The families we are looking for are the families

$$\mathcal{L}_A = \{F_\alpha^A : \alpha < \Omega,\ F_\alpha^A \neq \emptyset\}, \quad \mathcal{L}_B = \{F_\alpha^B : \alpha < \Omega,\ F_\alpha^B \neq \emptyset\}$$

Let us return to the proof of Theorem 2.

For any distinct points $x,y \in \mathbb{R}$ there exist compact intervals A, B with $x \in \text{Int } A$, $y \in \text{Int } B$ and $A \cap B = \emptyset$. Thus (by Lemma 2) there is a set $F_A \in \sigma(\mathcal{F}_0)$ which contains A, and $F_A \cap B = \emptyset$. From Theorem 1 it follows that condition (**) is sufficient.

The necessity of condition (**) follows from Lemma 1.

Theorem 2 gives a necessary and sufficient condition which is convenient to use but it concerns families of open intervals in $\mathbb{R}$. Examples 4 and 5 show that it is difficult to generalize this to other families and other spaces.

EXAMPLE 4: The family $\mathcal{K}_1 = \{R - \{x\} : x \in \mathbb{R}\}$ satisfies the condition

(***) for any distinct points $x,y \in \mathbb{R}$ there exists $K \in \mathcal{K}_1$ such that $x \in K$ and $y \notin K$

which is stronger than (**). $\mathcal{K}_1$ consists of open sets and generates the countable-cocountable structure on $\mathbb{R}$ which is essentially smaller than $\mathcal{B}_\mathbb{R}$.

EXAMPLE 5: The family $\mathcal{K}_2 = \{((-n,n) \times (-n,n)) - \{x\} : x \in \mathbb{R}^2,\ n \in \mathbb{N}\}$ consists of connected, bounded open sets, satisfies condition (***) but does not satisfy (*), (cf. Theorem 1) and therefore it is not a generator for $\mathcal{B}_{\mathbb{R}^2}$.

Now we shall consider families of convex sets and countable families.

THEOREM 3: Let X be a locally compact, σ-compact and perfectly normal linear topological space. If a family $\mathcal{A} \subset \mathcal{B}_X$ of open convex sets satisfies the condition:

(***) for any distinct points $x,y \in X$, there exists $G \in \mathcal{A}$ such that $x \in G$ and $y \notin G$,

then $\sigma(\mathcal{A}) = \mathcal{B}_X$.

Proof. Suppose we are given distinct points $x,y \in X$. It is enough to find a set $G \in \mathcal{A}$ such that $x \in G$ and $y \notin \bar{G}$. Let c be the midpoint of the interval joining the points x and y. By (***) there is a set $G \in \mathcal{A}$ such that $x \in G$ and $c \notin G$. This set is convex. If $y \in \bar{G}$, then the open interval joining x and y is contained in Int G ([3], p. 110). In particular $c \in \text{Int } G$, which gives a contradiction. Hence $y \in \bar{G}$. So the family $\mathcal{B}$ satisfies condition (*) from Theorem 1, which completes the proof.

Theorem 3 concerns locally compact and σ-compact linear topological spaces and hence finite dimensional spaces. (See [3], p. 62.) It is difficult to generalize this to infinite dimensional linear topological spaces.

EXAMPLE 6: Let B be the space of all bounded functions $f : \mathbb{R} \to \mathbb{R}$ with the metric $\rho(f,g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}$. Let $\mathcal{B} = \{f \in B : f(x) \in (a - \frac{1}{n}, a + \frac{1}{n})\} : x \in \mathbb{R}, a \in \mathbb{R}, n \in \mathbb{N}\}$.

The family $\mathcal{B}$ consists of open convex sets. Suppose that $A \in \sigma(\mathcal{B})$. There exists ([2], p. 24, Theorem D) a countable subclass $\mathcal{D}$ of $\mathcal{B}$ such that $A \in \sigma(\mathcal{D})$. So for uncountable many $y \in \mathbb{R}$ $\{f(y) : f \in A\} = \mathbb{R}$. Thus $\sigma(\mathcal{B})$ does not contain $\{f \in B : |f(x)| < 1\}$.

THEOREM 4: Let X be a topological space and let $\mathcal{H} = \{H_n : n \in \mathbb{N}\}$ be a countable family of compact subsets of X. If the family $\mathcal{H}$ satisfies the condition:

(***) for any distinct points $x,y \in X$ there is $n \in \mathbb{N}$ such that $x \in H_n$ and $y \notin H_n$,

then $\sigma(\mathcal{H}) = \mathcal{B}_X$.

Proof: Let $\mathcal{H}^*$ be the collection of all finite intersections of sets from $\mathcal{H}$. It is clear that $\mathcal{H}^*$ is countable and closed under finite intersections. We shall show that $\mathcal{H}^*$ is a pseudo-basis in X; i.e. for any $V \in \text{top } X$ and $x \in V$, there is a set $H \in \mathcal{H}^*$ such that $H \subset V$.

Suppose we are given $V \in \text{top } X$, $x \in V$. From condition (***) it follows that $\{x\} = \cap \{H \in \mathcal{H} : x \in H\} = \cap \{F \in \mathcal{H}^* : x \in F\}$. There exists a decreasing sequence $\{F_n\}_{n \in \mathbb{N}}$ of sets of $\mathcal{H}^*$ with $\{x\} = \bigcap_{i=1}^{\infty} F_i$. It is enough to show that there is a positive integer n such that $F_n \subset V$. Suppose it is not so. Then $F_n - V \neq \emptyset$ for each $n \in \mathbb{N}$. Each set $F_n - V$ being a closed subset of the compact set F_1 is a compact set. The sequence $\{F_n - V\}_{n \in \mathbb{N}}$ is a decreasing sequence of compact sets. Thus $\bigcap_{i=1}^{\infty} (F_i - V) \neq \emptyset$. However

$$\bigcap_{i=1}^{\infty} (F_i - V) = (\bigcap_{i=1}^{\infty} F_i) - V = \{x\} - V = \emptyset .$$

This contradiction proves that $\mathcal{N}^*$ is a pseudo-basis in X.

Each open set U in X can be represented as a union of sets from the countable family $\mathcal{N}^*$. So $U \in \sigma(\mathcal{N}^*)$. Therefore $\mathcal{B}_X = \sigma(\mathcal{N}^*) = \sigma(\mathcal{N})$.

THEOREM 5: Let X be a topological space and let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a countable family of open, relatively compact sets. If the family $\mathcal{U}$ satisfies the condition:

(***) for any distinct points $x, y \in X$ there is $n \in \mathbb{N}$ such that $x \in U_n$ and $y \notin U_n$,

then $\sigma(\mathcal{U}) = \mathcal{B}_X$.

Proof: Let $H_n = X - U_n$ for each $n \in \mathbb{N}$. For each $x \in X$, $\{x\} = \cap \{H_n : x \in H_n\}$. By our assumption there is a $k \in \mathbb{N}$ such that $x \in U_k$. Let $A = \bar{U}_k$. The family $\{H_n \cap A : n \in \mathbb{N}\}$ satisfies the assumptions of Theorem 4. Let W be a neighborhood of x. $W \cap U_k$ is a neighborhood of x also. From the proof of Theorem 4 it follows that there exists a set F, $F \subset W$ belonging to the algebra generated by $\{H_n \cap A : n \in \mathbb{N}\}$ or equivalently there is a set H, $H \cap A \subset W$, belonging to the algebra generated by $\{H_n : n \in \mathbb{N}\}$. (See [2], p. 25.)

That is why $H \cap U_k \subset H \cap A \subset W$ and $H \cap U_k$ belongs to the algebra generated by $\{U_n : n \in N\}$. This algebra is countable ([2], p. 23). Thus as in the proof of Theorem 4, $\mathcal{B}_X = \sigma(\{U_n : n \in \mathbb{N}\})$.

Lemma 1 and Theorems 1-5 prove that there is an essential relationship between separating fmailies and σ-algebras of Borel sets. The σ-algebra $\mathcal{B}_X$ is the smallest σ-algebra which satisfies one of conditions (*), (**) or (***). The following questions remain open:

1. May we replace condition (***) by condition (**) in Theorems 3, 4 and 5 and in this way formulate a necessary and sufficient condition?

2. Can we find a countably generated, separating σ-algebra which is contained in the σ-algebra $\mathcal{B}_{\mathbb{R}}$ (and not equal to $\mathcal{B}_{\mathbb{R}}$)?

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