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DERIVATIVES AND N.S.A.(NON STANDARD ANALYSIS)

1.- What you may say about classical derivatives -

When you study the possible generalizations of a derivative of a mapping f from a collection A into a collection B, you need to avoid as much as you can the particular properties of the specific structures of A and B.

Let $\phi(x, a)=[f(x)-f(a)]/(x-a)$ (1) be a mapping from A^2 into a collection C, the derivative f'(a) at the point a of A is classically defined by $f'(a)=\lim \phi(x,a)$ when $x \to a$.

For instance, if f'(a) is a vector derivative, A is a field, B, an n-dimensional vector space and C, a p-dimensional ($p \leqq n$) vector space.

The meaning of the expression "$x \to a$" is more accurate if we consider a mapping d_A from A^2 into an ordered structure (E, <, 0=infE), dense in 0 $(u \in E)(\exists v \in E)\ 0 < v < u$ (u ε E) means: for any u in E .

Hence, "$x \to a$" means $(u \in E)(\exists x \in A)\ 0 < d_A(x,a) < u$

We write "$x \underset{E'}{\to} a$" if $E' \in$ cofE (cofinality of E) $(u \in E)(\exists v \in E')\ 0 < v < u$

and if $(u \in E)(\exists x \in A)\ d_A(x,a) \in E'$ and $0 < d_A(x,a) < u$ (2)

generalized derivatives - If d_C is a mapping from C^2 into E,

$D_{\phi,E_A,E_C,d_A,d_C} f(a)=f'(a) \in C$ is a derivative of f(x) at $a \in A$ if $E_A, E_C \in$ cofE and $(v \in E)(\exists u \in E)$ if $d_A(x,a) \in E_A$ and $0 < d_A(x,a) < u$ then

$d_C[\phi(x,a),f'(a)] \in E_C$ and $0 < d_C[\phi(x,a),f'(a)] < v$

is a mapping, not always defined by (1) but depending on the wanted generalization of the derivative.

(E,<, 0) is the only structure used.

2.- What they say about N.S.A. (N.S.A. terminology) -

2.1 - Non standard extension of (E, <) - The order < induces an order <' on a collection ${}^XE$ of mappings from a collection X into E by

$(f,g \in {}^XE)$ f <' g iff $(x \in X)\ f(x) < g(x)$ but even if E is totally ordered, many elements f and g are not comparable.

The number of comparable elements can be increased if

$(f,g \in {}^XE)$ f <' g iff $\{x \in X : f(x) < g(x)\} \in F$, F being a non principal filter on X

A <u>non standard extension</u> of E is the collection ${}^*E = {}^X E/F$ such that ${}^*f \in {}^*E$ iff ${}^*f = \{ g \in {}^X E: \{x: g(x)=f(x)\} \in F$. Hence,

${}^*f \; {}^*\!< {}^*g$ iff $(f \in {}^*f)(g \in {}^*g) \; \{ x \in X: f(x) < g(x)\} \in F$

2.2 - <u>Standard elements of ${}^*E$</u> - Let u be an element of E. ${}^*u \in {}^*E$ iff ${}^*u = \{f \in {}^X E: \{ x \in X: f(x)=u\} \in F \}$ (${}^{st}E$: collection of standard elements of ${}^*E$)

2.3 - <u>Halo of ${}^*u$</u> (a halo is a <u>monad</u> in [1] and [2]) -

The halo of ${}^*u$ is the collection $Hal({}^*u)$ such that ${}^*f \in Hal({}^*u)$ iff $(v \in {}^{st}E) \; \{ x \in X: \inf(u,v) < f(x) < \sup(u,v)\} \in F$ with $\inf(u,v)=\inf(v,u)=u$ and $\sup(u,v)=\sup(v,u)=v$ if $u < v$

<u>Superior</u> and <u>inferior halo</u>, $Hal^+({}^*u) = \{ {}^*f \in Hal({}^*u): u < f(x)\}$ and $Hal^-({}^*u) = \{ {}^*f \in Hal({}^*u): f(x) < u \}$ can also be defined.

A <u>near standard</u> element is an element in $Hal({}^*u)$ (${}^*u \in {}^{st}E$).

If u is an origin of E ($(u \in E)$ $u < 0$ or $0 < u$) then $Hal({}^*0)$ is the collection of <u>infinitesimals</u> of ${}^*E$.

2.4 - <u>Main difference between classical analysis (C.A.) and non standard analysis (N.S.A.)</u> -

C.A. uses properties of density: $(u \in E)(\exists v \in E)$ if $u < a$ then $u < v < a$ (fig.1)

N.S.A. uses properties of near standard elements :

if $u \in Hal^-(a)$ then $(v \in {}^{st}E)$ if $v < a$ then $v < u < a$ (fig.2)

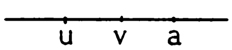

fig. 1 $(u,v,a \in {}^{st}E)$

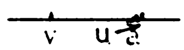

fig.2 $(v,a \in {}^{st}E)$

2.5 - <u>Infinite elements</u> - An element ${}^*f$ is an <u>infinite positive element</u> iff $(f \in {}^*f)(u \in E) \{x \in X: u < f(x)\} \in F$. Negative infinite elements can be defined and even infinite elements neither positive nor negative if F is not an ultra-filter.

2.6 - <u>Non standard extension of a mapping from $(A, <)$ into $(B, <)$</u> -If u and v are mappings from X into A and B, defining elements ${}^*u \in {}^*A = {}^X A/F$ and ${}^*v \in {}^*B = {}^X B/F$, the non standard extension of f is ${}^*f$ such that ${}^*f({}^*u) = {}^*v$ if $\{ x: f(u(x))=v(x) \} \in F$.

2.7 - <u>Transfer principe</u> - The definition of a derivative with a mapping ϕ needs structures on A, B and C with, at least, two binary operations (see 1: "classical derivatives"). For instance, if an additive operation + is defined by a mapping from A^2 into A, an additive operation ${}^*+$ can be defined on ${}^*A$ by ${}^*a \; {}^*\!+ \; {}^*a' = {}^*a''$ and $\{x \in X: a(x)+a'(x)=a''(x)\} \in F$.

If $^*a, ^*a' \in {}^{st}A$, then $^*+$ determine on ^{st}A, a structure isomorphic to the structure determined by + on A.

The generalization of this property to any relation leads to the transfer principle used in N.S.A.

3.- **What you can say about non standard derivatives** -

A non standard derivative $^*f'$ of a mapping f from A into B is given by non standard extensions $^*f$ of f from $^*A$ into $^*B$ and $^*\phi$ from A^2 into $^*C$ of the mapping ϕ defining f'.

We have if "($\exists f'(a) \in C$) if $a' \in Hal(^*a)$ then $^*f'(a') \in Hal[^*f'(^*a)]$ " then $^*f'(^*a)$ is the non standard derivative.

For instance, we can say that $^*f$ is continuous at $^*a$ when if $a' \in Hal(^*a)$ then $^*f(a') \in Hal[^*f(^*a)]$. Hence $^*f$ is the non standard derivative of order zero of $^*f$ at the point $^*a$.

It may be prooved that if a classical derivative f'(a) does exist, the non standard derivative is $^*f'(a) \in {}^{st}C$. But a non standard derivative may exist even if f'(a) doesn't.

For instance, if a is an isolated point of A $((\exists a_p, a_s \in A)(a' \in A)\ a' < a_p$ or $a_s < a'$ or $a'=a_p, a_s, a)$. In that case, if $^*a' \in {}^*A$ is defined by $a' \in {}^XA$ such that $(\exists x_0 \in X)\{x \in X: Im\{a'(x):x_0 < x\} = \{a_p, a, a_s\}\} \in F$ then $a' \in Hal(a)$ and it may exist f'(a) such that $^*f'(a') \in Hal[^*f'(^*a)]$.

N.S. Analysts may also define $^*f'(^*a)$ when $^*a$ or $^*f'(^*a)$ are infinite elements of $^*A$ or $^*C$ and n-non standard derivatives $^{n*}f'(^*a)$ if they consider iterated non standard extensions $^n{}_s{}^*E$ of any collection E.

4.- **Comments** - The concept of non standard derivative is useful only with a good knowledge of N.S.A., especially when infinite elements are needed: in that case, mappings with an infinite number of values are used. N.S. Analysts use the concurrent theoreme: "infinite mappings" are defined by a collection of "finite mappings" : the concurrent relations .

The sets can be defined as elements of iterated non standard extensions $^n{}_s{}^*E_0 = {}^*(^{n*}E_0)$ of a boolean structure on $E_0 = \{0,1\}$. But it does exist collections of elements that cannot be defined in that way : external sets .

Nevertheless, it could be interesting to understand what N.S. Analysts say about non standard derivatives.

REFERENCES

[1] Martin Davis, "Applied non standard analysis" John Wiley (1977)

[2] Abraham Robinson, "Non standard analysis" Studies in Logic, North-Holland (1966)