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Peano curves and continuous functions whose all level sets are perfect

In the present paper we give an analytic representation of the Peano curve (see fig. 1) and we prove that its graph fills a square and its components are continuous functions whose all level sets are perfect. These two components seem to be simpler than the example given in [2](p.223). Note that in 1930, Nina Bary ([1],p. 640) mentioned the existence of functions whose all level sets are perfect: "On obtient de telles fonctions en considérant l'une des composantes d'une courbe péanienne convenablement construite". In 1939 Gillis [3] constructed a nonconstant continuous function on [0,1] with the property that every level set is perfect, but the author has not seen his paper (see [2],p.214).

Let $f_1, f_2 : [0,1] \longrightarrow [0,1]$ be two continuous functions defined as follows: for each $t \in [0,1]$, $t = \sum_{i=1}^{\infty} t_i/3^i$, $t_i \in \{0,1,2\}$, $t_0=0$,

$$f_1(t) = a_1/3 + \sum_{i=2}^{\infty} h(t_0)\cdot h(t_2)\cdot \ldots \cdot h(t_{2i-4})\cdot(a_{2i-1}/3^i) \quad \text{(see fig.2)}$$

$$f_2(t) = a_2/3 + \sum_{i=2}^{\infty} h(t_1)\cdot h(t_3)\cdot \ldots \cdot h(t_{2i-3})\cdot(a_{2i}/3^i) \quad \text{(see fig.3)};$$

where $a_1 = t_1$; $a_i = t_i$ if t_{i-1} is even; $a_i = 3-t_i$ if t_{i-1} is odd; $h(t_i) = +1$ if t_i is even; $h(t_i) = -1$ if t_i is odd.

Let $F:[0,1]\longrightarrow[0,1]\times[0,1]$, $F(t) = (f_1(t),f_2(t))$. From fig.2 and fig.3 we can see that F is in fact the Peano curve of fig.1.

Let $A(t) = \{i : t_{2i-1} \text{ is odd}\} = \{i_1,i_2,i_3,\dots\}$, $i_k< i_{k+1}$, $k = 1,2,\dots$, $i_0 = 1$. If $i\in\{1,2,\dots,i_1-1\}\cup\{i_k+1,\dots,i_{k+1}-1\}$, $k = 1,2,\dots$, then t_{2i-1} is even, hence $a_{2i} = t_{2i}$ and $h(t_{2i-1}) = +1$. Also, $a_{2i_k} = 3-t_{2i_k}$ and $h(t_{2i_k-1}) = -1$. We have

$$f_2(t) = \sum_k (-1)^k \sum_{i=i_k+1}^{i_{k+1}} a_{2i}/3^i = \sum_k (-1)^k \Big(\sum_{i=i_k+1}^{i_{k+1}-1} t_{2i}/3^i +$$

$$(3-t_{2i_1})/3^{i_1}\Big) = \sum_k (-1)^k \sum_{i=i_k}^{i_{k+1}-1} t_{2i}/3^i + \sum_k (-1)^k(1/3^{i_{k+1}-1}) =$$

$$\sum_{\substack{i=i_p\\ p=\text{even}}}^{i_{p+1}-1} t_{2i}/3^i + \sum_{\substack{i=i_p\\ p=\text{odd}}}^{i_{p+1}-1} (2-t_{2i})/3^i .$$

Let $B(t) = \{i : t_{2i} \text{ is odd}\} = \{j_1,j_2,j_3,\dots\}$, $j_k < j_{k+1}$, $k = 1,2,\dots$, $j_0 = 0$. Then by an analogous argumentation we have:

$$f_1(t) = \sum_{\substack{i=j_p+1\\ p=\text{even}}}^{j_{p+1}} t_{2i-1}/3^i + \sum_{\substack{i=j_p+1\\ p=\text{odd}}}^{j_{p+1}} (2-t_{2i-1})/3^i .$$

<u>Theorem.</u> a) <u>The function</u> F <u>is a surjection.</u>
b) <u>Let</u> $x_o,y_o\in[0,1]$. <u>Then</u> $\{t : f_1(t) = x_o\}$ <u>and</u> $\{t : f_2(t) = y_o\}$ <u>are perfect sets.</u>

<u>Proof.</u> a) Let $(x_o,y_o)\in[0,1]\times[0,1]$ with $x_o = \sum_{i=1}^{\infty} x_i/3^i$ and $y_o = \sum_{i=1}^{\infty} y_i/3^i$, $x_i,y_i\in\{0,1,2\}$. Let $\{i_k\}_k$ be the increasing

sequence of all i such that $x_i = 1$ and let $\{j_k\}_k$ be the increasing sequence of all i such that $y_i = 1$. Let $t_o = \sum_{i=1}^{\infty} t_i/3^i$ such that:

for $i \in \{j_p+1,\dots,j_{p+1}\}$ we have $t_{2i-1} = x_i$ if p is even and $t_{2i-1} = 2-x_i$ if p is odd; and

for $i \in \{i_p,\dots,i_{p+1}-1\}$ we have $t_{2i} = y_i$ if p is even and $t_{2i} = 2-y_i$ if p is odd.

Then $A(t_o) = \{i_1,i_2,\dots\}$, $B(t_o) = \{j_1,j_2,\dots\}$ and $F(t_o) = (x_o,y_o)$.

b) Let $t_o \in \{t : f_2(t) = y_o\}$, $t_o = \sum_{i=1}^{\infty} t_i/3^i$, $t_i \in \{0,1,2\}$.

1) If $A(t_o) = \{i_1,i_2,\dots\}$ is infinite let

$$u_n = \sum_{i=1}^{2i_{2n}-1} t_i/3^i + \sum_{\substack{i=2n+p \\ p=\text{even}}}^{i_{2n+p+1}-1} t_{2i}/3^{2i} + \sum_{\substack{i=2n+p \\ i=\text{odd}}}^{i_{2n+p+1}-1} (2-t_{2i})/3^{2i}.$$

Then $A(u_n) = \{i_1,\dots,i_{2n}\}$, $u_n \longrightarrow t_o$ and $f_2(u_n) = y_o$.

2) If $A(t_o) = \{i_1,\dots,i_p\}$ let

$$u_n = \sum_{i=1}^{2i_p+2n} t_i/3^i + 1/3^{2i_p+2n+1} + \sum_{i=i_p+n+1}^{\infty} t_{2i}/3^{2i}.$$

Then $A(u_n) = \{i_1,i_2,\dots,i_p,i_p+n+1\}$, $u_n \longrightarrow t_o$ and $f_2(u_n) = y_o$.

By 1) and 2), $\{t : f_2(t) = y_o\}$ is perfect. For f_1 the proof is analogous.

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References

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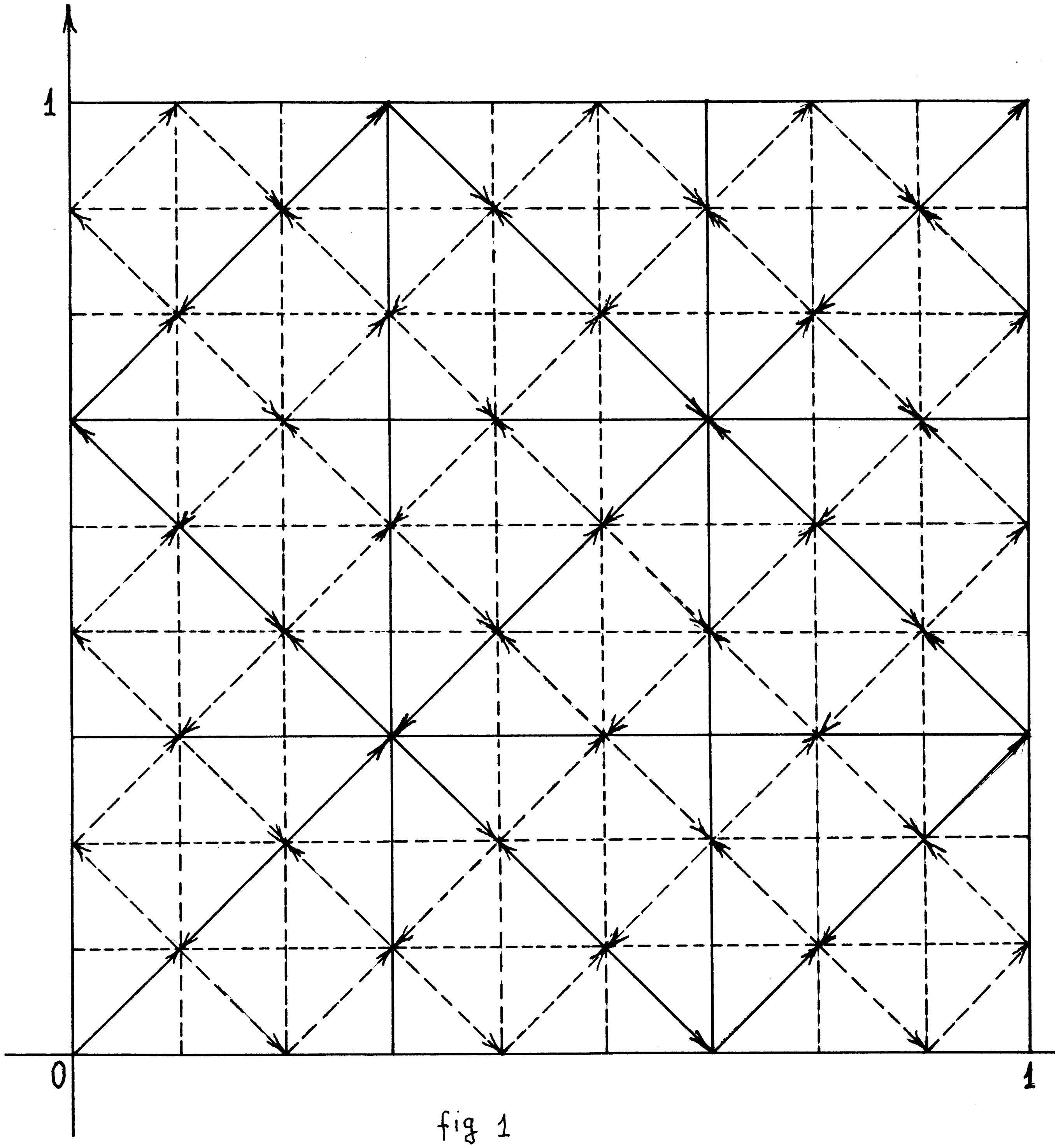

fig 1

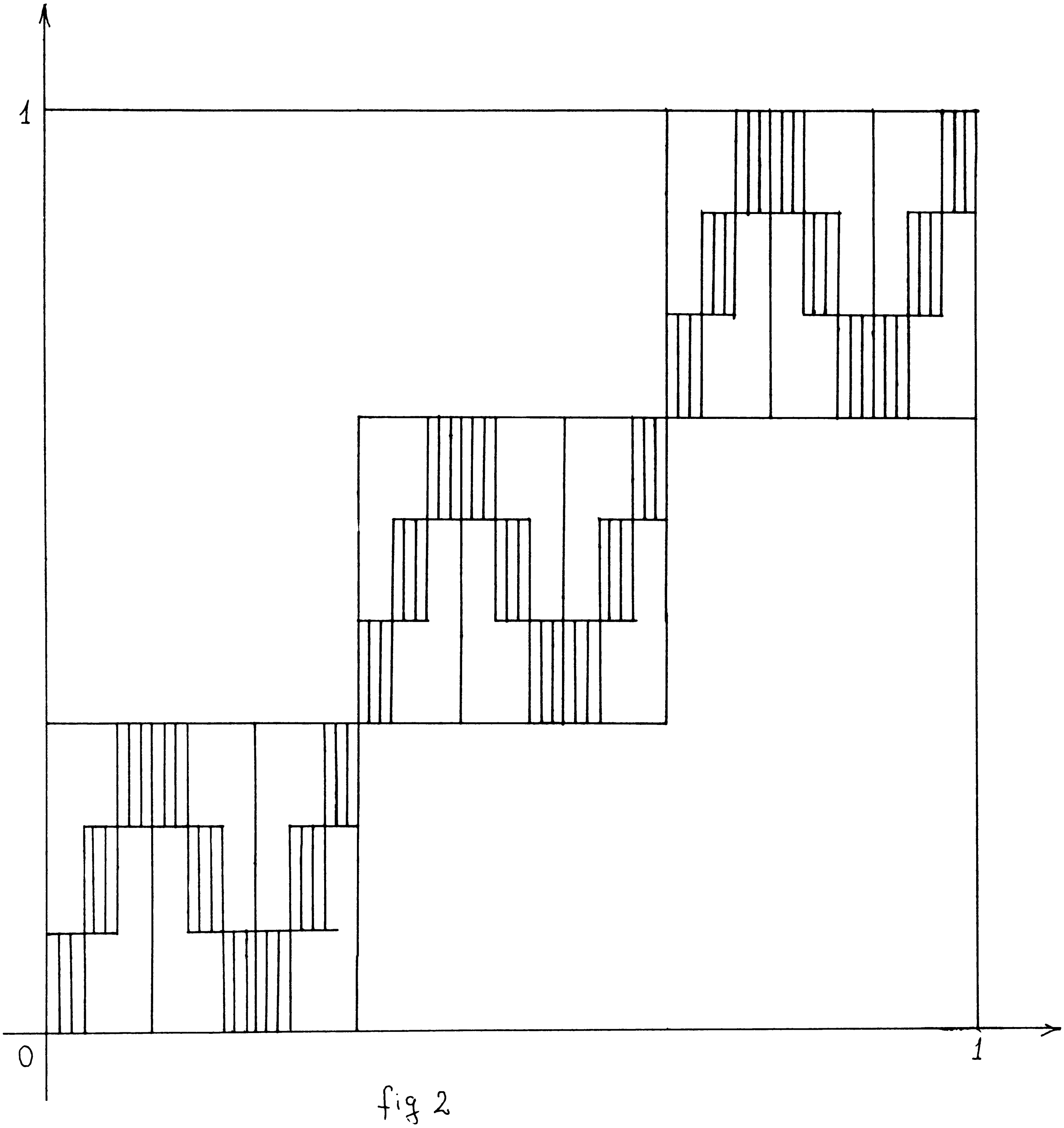

fig 2

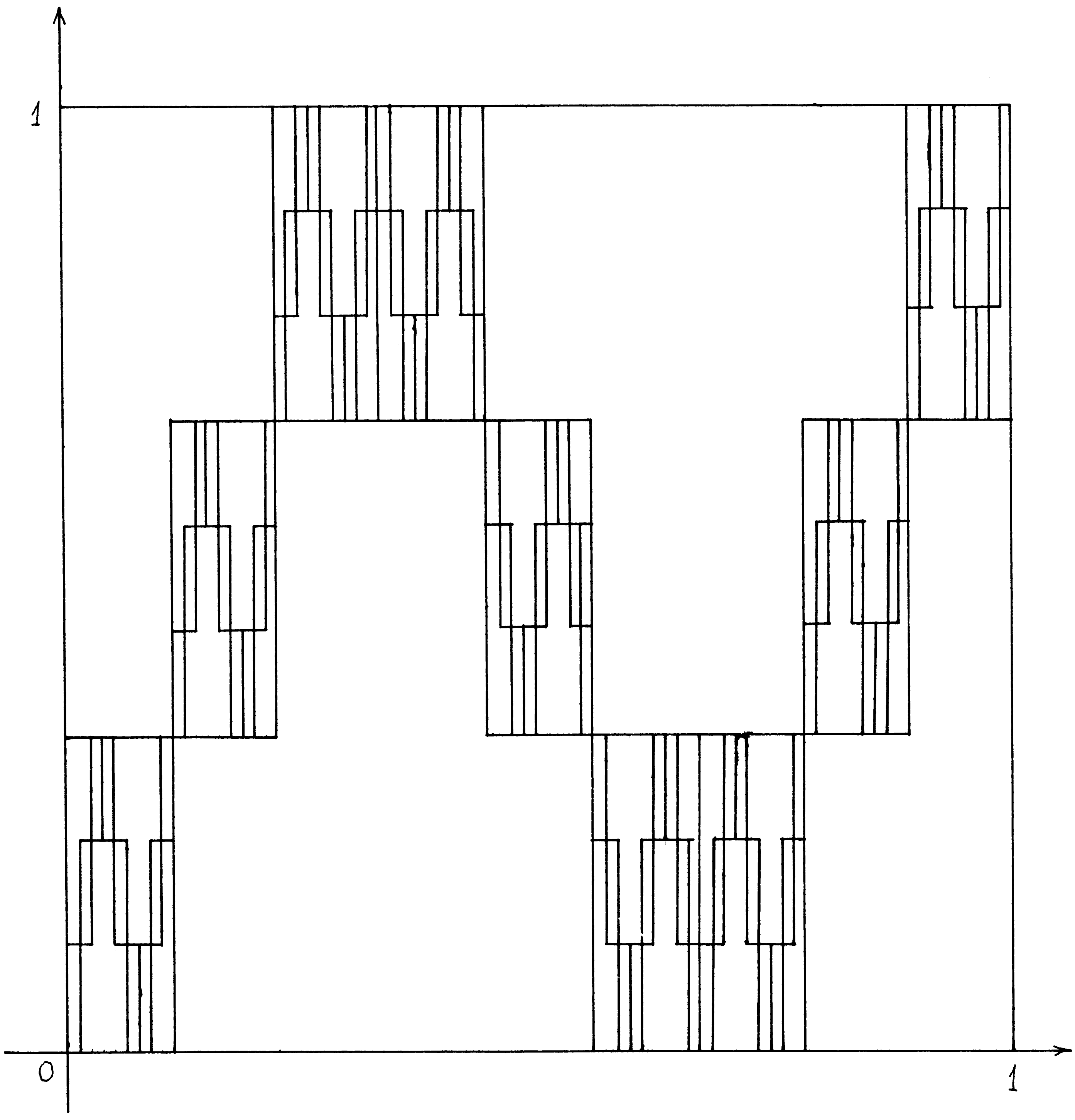

fig 3