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On Lusin's condition (N) and some of its modifications

A real-valued function of a real variable is said to satisfy Lusin's condition (N) on a set E if its image of each null subset of E is also a null set. Clearly, every linear function satisfies the condition (N). Mazurkiewicz had shown that there exists a continuous function f which satisfies the condition (N) on an interval while f + g does not satisfy the same condition for every non-constant linear function g. Viewing this non-additive property as a kind of irregularity, we are led to seek conditions which are "near" to the condition (N) but do not have the same kind of irregularity. [Notations without explanations here will be those used in Saks' book.]

Banach's condition (S) can be thought as one which is nearest to the condition (N). A function f is said to satisfy the condition (S) on E if for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f[H]| < \epsilon$ whenever $H \subset E$ and $|H| < \delta$. Note that the class S [of all the functions satisfying the condition (S)] and the class N have the following relations (mainly due to Bary)

(1) $$CS \subsetneq CN \subsetneq CS + CS \subsetneq CS + CS + CS = C$$

on any interval, where C denotes the class of all the continuous functions, AB denotes the intersection of A and B, A + B denote the set of all a + b with $a \in A$, $b \in B$. Since $CS \subset CN$ and since

Mazurkiewicz's function is also in CS, one sees that the condition (CS) has the same non-additive property as (CN): $CS + L \neq CS$, where L denotes the class of all the linear functions. Similarly $CS + CS + L \neq CS + CS$. However, we have $CS + CS + CS + L = CS + CS + CS$ since both sides are the same as C, which is additive. Thus (CS+CS+CS) does not have the irregular non-additive property as the other conditions mentioned up to now. Unfortunately, the property (CS+CS+CS) is (C) and is too far away from the important property (N), and hence is not interesting for our concern here. We ask whether there exists a condition (A) between (CN) and (CS+CS) which has the additive property $A + L = A$. The answer is yes. In fact Foran's condition (M) has such a property. A continuous function is said to satisfy the condition (M) if it is AC on each set E on which the function is VB. Foran had shown that $CN \subsetneq M$ and also that every function in M is differentiable at every point of a set of positive measure in every interval. From the last property it follows that one also has $M \subset CS + CS$ by a theorem due to Bary. Furthermore, it is easily checked that $M + L = M$ (cf. also Ene).

For clarity and for possible further consideration, we give the following definition.

<u>Definition</u>. Let (F), (G), (U), (V) be certain function theoretic conditions. The condition (U) is called an additive extension of

(F) by (G) if the corresponding classes of functions satisfy the relations $U \supset F$ and $U + G = U$. The condition (V) is called an additive restriction of (F) by (G) if $V \subset F$ and $V + G = V$.

For any (F) and (G), denote

$$AE[F,G] = \cap\{U: (U) \text{ is an additive extension of } (F) \text{ by } (G)\}$$

$$AR[F,G] = \cup\{V: (V) \text{ is an additive restriction of } (F) \text{ by } (G)\}.$$

Then one shows easily that (AE[F,G]) is an additive extension of (F) by (G), and is the smallest of such extension. Similarly, (AR[F,G]) is the largest additive restriction of (F) by (G).

With the terminologies above, one of the results discussed up to now can be stated as follows: the condition (M) is an additive extension of (CN) by (L). However, we conjecture that (M) is not the smallest such extension, i.e. $M \neq AE[CN,L]$. Further investigations to obtain descriptive characterizations for the smallest extension (AE[F,G]) and the largest restriction (AR[F,G]) might be hard but challenging.

To discuss additive restriction of (CN) by (L), let us recall the following relations:

(2) $AC \subsetneq ACG \subsetneq CSG \subsetneq CN \subsetneq CS + CS.$

(3) $AC \subsetneq CS \subsetneq CSG \subsetneq CN.$

Here (SG) denotes the generalized condition of (S) in the sense that a function satisfies the condition (SG) on a set E if E is the union of countably many sets on each of which the function satisfies the condition (S). Note that one can not combine (2) and (3) into one string because ACG ~ CS $\neq \phi$ and CS ~ ACG $\neq \phi$.

As mentioned earlier at the beginning that (CS) is not an additive restriction of (CN) by (L), neither is the condition (CSG). It is also clear that each of the AC, ACG_*, ACG is an additive class and contains L, so that each of these is clearly an additive restriction of (CN) by L. We do not know whether there will be a condition between (CSG) and (CN) or between (AC) and (CS) which is an additive restriction of (CN) by L. However, there are conditions between (ACG) and (CSG) which are additive restrictions of (CN) by (L), or even of (CN) by (ACG). These are mainly of works by Foran, Ene, and Iseki. For example, Foran's condition ($\mathcal{F}$) is known to be additive and between (ACG) and (CN). It is not difficult to see that in fact ($\mathcal{F}$) is between (ACG) and (CSG).

Since the Hausdorff measures have been a revived interesting recently, we would like to mention here some other conditions "near" the condition (N), using the concepts of length of sets as given in Saks.

For each pair of positive numbers α, β, a function ℓ is said to satisfy the condition (N_α^β) on a set E if

$$\Lambda_\beta(\ell[Z]) = 0 \text{ whenever } Z \subset E \text{ and } \Lambda_\alpha(Z) = 0.$$

Similarly, (S_α^β) can be defined, and lots of non-trivial questions arise. Answers to some of the possible problems related to the concepts of additive extensions or restrictions might have important applications to various theories of integrations. We end this note by listing some (not all) of the references quoted in this note.

[1] V. Ene, Monotonicity theorems, Real Analysis Exchange 12 (1986-87), 420-454.

[2] J. Foran, An extension of Denjoy integral, Proc. Amer. Math., 49 (1975), 359-365.

[3] K. Iseki, An attempt to generalize the Denjoy integration, Natural Science, Ochanomizu Univ., 34 (1983), 19-33.

[4] S. Saks, Theory of the Integral, Warsaw (1937).