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A GENERAL RIEMANN COMPLETE INTEGRAL IN THE PLANE

In /P1/ W.F.Pfeffer defines a multidimensional Riemann type integral such that the divergence of any vector field continuous in a compact interval and differentiable in its interior is integrable, and the integral equals the flux of the vector field out of the interval. In section 7.4. he mentions the possibility of a definition of an integral using special partitions. This integral is a natural, simple alternative of his integral. In our paper /B/ we solve the problem of the existence of special partitions in the plane. Unfortunately we were unable to use our method for higher dimensions. But this two dimensional case may be also of interest because there are numerous applications of the divergence theorem in the plane.

Definitions of Riemann type integrals are based on the ideas of Henstock and Kurzweil (cf./H1/,/H2/ and /K1/). Mahwin's integrals in /M1/ or /M2/ are suitable generalizations for the divergence theorem but they lack the additivity, which one expects from an integral.

In the first section of /B/ we prove the existence of special partitions. In the second section we give the definition of our general Riemann complete (GRC) integral and then we formulate some of its properties. We prove the additivity of this integral and we also show that this integral is more general then the usual RC integral or Mahwin's integral. We state the almost everywhere differentiability of the integral functions (Theorem 2) and the divergence theorem (Theorem 3) without proof. They are stated in /P1/ for the integral defined there and their proof applies to our integral as well. We remark that all lemmas, propositions, theorems and corollaries in sections 3-6 in /P1/ can be proved for the GRC integral. Finally we prove that this GRC integral integrates the derivatives of functions of intervals.

By the plane we mean $\mathbb{R}^2$. The term measurable means Lebesgue measurable and $|A|$ denotes the Lebesgue measure of A. By an interval we mean a compact non-degenerate rectangle with sides parallel to the coordinate axes.
The unit square, $[0,1]\times[0,1]$ is denoted by I.
DEFINITION 1. Let $\delta : I \to (0,+\infty)$ be given. We say that an interval A is special if there is a vertex V of A such that $A \subset \{x:\ \mathrm{dist}(V,x) < \delta(V)\}$.

DEFINITION 2. An interval A is d-regular if $\min(a,b)/\max(a,b) > d$ where a and b are the lengths of the sides of A.

THEOREM 1. Let $\delta: I \rightarrow (0,\infty)$ be given. Then I can be partitioned into finitely many non-overlapping $5\cdot 10^{-3}$-regular special intervals.

We remark that the constant $5\cdot 10^{-3}$ obviously can be improved, although we do not know whether it can be close to 1.

DEFINITION of the GRC integral.

Let A be an interval in the plane. A partition of A is a collection

$P = \{ (A_1,x_1),\dots,(A_p,x_p) \}$ where $A_1,\dots,A_p$ are non-overlapping subintervals of A,

$x_i \in A_i$, $i=1,\dots,p$, and $\bigcup_{i=1}^{p} A_i = A$.

Given a $\delta: A \rightarrow (0,\infty)$, we say that P is δ-fine whenever $\operatorname{diam}(A_i) < \delta(x_i)$ for $i=1,\dots,p$.

Given a number s, we say that P is s-regular special if P consists of s-regular special intervals.

In Theorem 1 we proved that for the interval I there exists an $s > 0$ such that for any $\delta: I \rightarrow (0,\infty)$ we can find a δ-fine s-regular special partition of I.

If f is a function on an interval A and

$P = \{ (A_1, x_1), \ldots, (A_p, x_p) \}$ is a partition of A, we let:

$$I(f,P) = \sum_{i=1}^{p} f(x_i) \cdot |A_i| \ .$$

DEFINITION 3 :A function f on an interval A is called (GRC) integrable in A if there is a real number J with the following property: given $\varepsilon > 0$, there is a $\delta : A \to (0, \infty)$ such that

$|I(f,P)-J| < \varepsilon$ for each δ-fine s-regular special partition of A when $0 < s < (5/2)\ 10^{-3}$.

Definition 3 makes sense when the set of partitions fulfilling the restrictions is not empty.

Any interval can be divided into finitely many 1/2-regular intervals. And these intervals can be transformed by a linear transformation into I.

Thus Theorem 1. provides a δ-fine s-regular special subpartition of the elements of the above divison when $2s < 5 \cdot 10^{-3}$.

REFERENCES

/B/Z.Buczolich, A general Riemann Complete Integral in the plane, (to appear)

/H1/R.Henstock, Theory of integration, Butterworths, London, 1963

/H2/-,A Riemann type integral of Lebesgue power, Canad.J.Math 20 (1968),79-87.

/K1/J.Kurzweil , Generalized ordinary differential equations and continuous dependence on a parameter, Czechoslovak Math.J.7 (1957),418-446

/M1/J.Mawhin, Generalized Riemann integrals and the divergence theorem for differentiable vector fields, in E.B.Christoffel, Birkhauser -Verlag, Basel,1981,704-714.

/M2/-,Generalized multiple Perron integrals and the Green Goursat theorem for differentiable vector fields, Czechoslovak Math J.31(1981),614-632.

/P1/W.F.Pfeffer,The divergence theorem, Trans.Amer.Math.Soc.,Vol.295,Number 2,665-685.

/P2/-,The multidimensional fundamental theorem of calculus (to appear)