A Remark on Localization and $(C, 1)$ Localization

CASPER GOFFMAN
Purdue University

For $n = 2$, localization by rectangular sums holds for $f \in HBV$. For any $\wedge BV \not\subset HBV$, there is an $f \in \wedge BV$ for which localization by square sums does not hold, [2].

For $n > 2$, localization by rectangular sums holds for all $f \in W_1^p, p > n-1$, but fails for some $f \in W_1^{n-1}$, [4]. By square sums, it holds for all $f \in W_1^{n-1}$ but, for every $p < n-1$, it fails for some $f \in W_1^p$, [1].

For $(C, 1)$ localization the difference between localization by square sums and by rectangular sums is great in contrast with the above facts. For $n \geq 2, (C, 1)$ localization by square sums holds for every $f \in L^p, p \geq n-1$, but for any $p < n-1$, there is an $f \in L^p$ for which $(C, 1)$ localization by square sums fails, [3]. However, there is no p such that $(C, 1)$ localization by rectangular sums holds for all $f \in L^p$, [5].

In this note, the matter is further elucidated using a slight modification of the technique in [3]. We consider $(C, 1)$ localization for particular sequences of n tuples of positive integers. For square sums the sequence is $\{(j, j, ..., j)\}$. For each real $r \geq 1$, we consider the sequence $\{(j, ..., j, j^{r_j})\}$, where j^{r_j} is the integer part, $[j^r]$, of j^r.

Let $T^n = [-\pi, \pi)^n$ be the n torus, let $\varepsilon > 0$, and $g_\varepsilon(x) = 1$, for $|x| > \varepsilon$, and $g_\varepsilon(x) = 0$, for $|x| < \varepsilon$. For each n tuple $(j_1, ..., j_n)$ of positive integers, let $K_{j_1,...,j_n}$ be the associated Fejer kernel. Then

$$K_{j_1,...,j_n}(y) = K_{j_1}(y_1), K_{j_2}(y_2), ..., K_{j_n}(y_n)$$

where $y = (y_1, ..., y_n)$.

In order to show that for some $f \in L^p$, $(C, 1)$ localization with respect to a sequence $\{(a_1(j), ..., a_n(j))\}$ of n tuples of positive integers does not hold it suffices to show that

$$I_j = \int_{T^n} |g_\varepsilon(y) K_{a_1(j),...,a_n(j)}(y)|^q dy$$

is unbounded as $j \to \infty$, where q is the conjugate to p.

For $r \geq 1$, let $a_i(j) = j, j = 1, ..., n-1$, and let $a_n(j) = j^{r_j}$. Then $a_n(j)$ is an integer. By substituting for g_ε and $K_{a_1(j),...,a_n(j)}$ their values, we have

$$I_j \geq Aj^{-q(n-1)-qr_j} \int_\varepsilon^\pi \left(\frac{\sin(j+1)\frac{y_1}{2}}{\sin\frac{y_1}{2}}\right)^{2q} dy_1 \int_0^\pi \left(\frac{\sin(j+1)\frac{y_1}{2}}{\sim\frac{y_1}{2}}\right)^{2q} dy_2 ...$$
$$... \int_0^\pi \left(\frac{\sin(j+1)\frac{y_{n-1}}{2}}{\sin\frac{y_{n-1}}{2}}\right)^{2q} dy_{n-1} \int_0^\pi \left(\frac{\sin(j^{r_j}+1)\frac{y_n}{2}}{\sin\frac{y_n}{2}}\right)^{2q} dy_n$$
$$\geq Aj^{-q(n-1)-qr_j+(2q-1)(n-2)+(2q-1)\cdot r_j}.$$

The sequence $\{I_j\}$ is unbounded if the limit of the sequence of powers of j is positive. This limit is

$$-q(n-1) - qr + (2q-1)(n-2) + (2q-1)r$$
$$= qn + qr - 3q - n + 2 - r$$

So we wish to know for which p we have

$$qn + qr - 3p - n + 2 - r > 0,$$

or

$$q(n + r - 3) > n - 2 + r$$

or

$$q > \frac{n-2+r}{n+r-3}.$$

$$\text{But } q = \frac{p}{p-1},$$
$$\text{so } \frac{p}{p-1} > \frac{n-2+r}{n+r-3}$$
$$-p > -n + 2 - r,$$
$$p < n + r - 2,$$

Accordingly, there is no $(C, 1)$ localization of the specified type for some $f \in L^p$, for every $p < n + r - 2$.

We now suppose r_j satisfies $j^{r_j} = [j^r] + 1$.

The question regarding the value of p for which $(C,1)$ localization for the sequence $(j,...,j,j^{r_j})$ of n tuples holds for all $f \in L^p$ reduces as in (3) to those p for which

$$\sup_{j,x} |\sigma_{j,...,j,j^{r_j}}(x, g_\varepsilon f)| \leq A\|f\|_p.$$

Now, by Hölder's inequality,

$$|\sigma_{j,...,j,j^{r_j}}(x, g\varepsilon f)| \leq \|f\|_p J_j^{\frac{1}{q}}$$

where

$$J_j = \int_{T^n} |g_\varepsilon(y) K_{j,...,j,j^{r_j}}(x-y)|^q dy.$$

So

$$J_j \leq A j^{-q(n-q)-qr_j+(2q-1)(n-2)+(2q-1)r_j}.$$

The limit of the power of j is nonpositive if

$$p \geq n + r - 2.$$

Accordingly, there is $(C,1)$ localization of the specified type for every $f \in L^p$ for $p \geq n + r - 2$.

It also follows that $(C,1)$ localization fails with respect to the sequence $\{(j,j,...,j,j^j)\}$ for some $f \in L^p$, for every p.

We combined the above in the theorem.

THEOREM. *$(C,1)$ localization holds for every $f \in L^p$, for all $p \geq n-1$ for sequences of n tuples satisfying a parameter of regularlity.*

For any $r > 1, (C,1)$ localization holds with respect to the sequence $\{(j,...,j,j^{r_j})\}$ for every $f \in L^p$ for every $p \geq n+r-2$ and fails for some $f \in L^p$ for every $p < n+r-2$.

For any $p, (C,1)$ localization fails for some $f \in L^p$, with respect to the sequence $\{(j,...,j,j^j)\}$ of n tuples.

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West Lafayette, Indiana