R.M. Shortt, Department of Mathematics, Wesleyan University, Middletown, Connecticut 06457, U.S.A.

SETS WITH THE COUNTABLE CHAIN CONDITION

§0. Definitions

Let X be a subset of the real line $\mathbf{R}$. Let $\mathcal{B}(X)$ be the Borel σ-algebra of X, i.e.

$$\mathcal{B}(X) = \{B \cap X : B \text{ Borel subset of } \mathbf{R}\}.$$

Two subsets X_1 and X_2 of $\mathbf{R}$ are <u>Borel-isomorphic</u> if there is a one-one correspondence $f : X_1 \to X_2$ such that $B \in \mathcal{B}(X_1)$ if and only if $f(B) \in \mathcal{B}(X_2)$. Given an uncountable $X \subseteq \mathbf{R}$, define

$$t(X) = \{Y \subseteq \mathbf{R} : X \text{ and } Y \text{ Borel-isomorphic}\},$$

the <u>isomorphism type</u> of X. If X is countable, write $t(X) = 0$. Put

$$S = \{t(X) : X \subseteq \mathbf{R}\}.$$

The set S has both an algebraic and an order structure. Define a relation $\leq$ on S by declaring $t(X_1) \leq t(X_2)$ if either

1) X_1 is countable

or

2) there is a Borel-isomorphism of X_1 onto a Borel subset of X_2.

<u>0.1 Fact</u>: The relation $\leq$ partially orders the set S.

Given t_1 and t_2 in S, let $X_1 \subseteq (0,1)$ and $X_2 \subseteq (1,2)$ be such that

$t_1 = t(X_1)$ and $t_2 = t(X_2)$. Then put $t_1 + t_2 = t(X_1 \cup X_2)$. A similar method defines $t_1 + t_2 + \cdots$ for any sequence t_n in S. Write

$$nt = t + \cdots + t \qquad \text{(n times)}$$

$$\omega t = t + t + \cdots .$$

Thus, the elements to S have properties analogous to those of cardinal numbers. Indeed, (S,+,0) as defined above is a cardinal algebra in the sense of Alfred Tarski [6]. In fact, some of the following results can be derived using his theory. Let us simply note

0.2 Fact: Under the operation + and order $\leq$, the set S becomes a commutative, ordered semi-group with identity element 0.

0.3 Fact: The set S has cardinality 2^c. A subset R of S is bounded above if and only if $\text{card}(R) \leq c$.

0.4 Fact: The partially ordered set S is not a lattice. It contains two elements with no infimum.

Say that a set $X \subseteq \mathbf{R}$ is measurably rigid if whenever $f : X \to X$ is a Borel-isomorphism of X onto itself, then $\{x : f(x) \neq x\}$ is countable. The existence of uncountable measurably rigid sets was demonstrated in [1] (see also [5]). Call $t \in S$ rigid if $t \neq 0$ and $t = t(X)$ for some measurably rigid $X \subseteq \mathbf{R}$.

§1. Spaces with c.c.c.; complete subsets of S

A set $X \subseteq R$ will satisfy the (measurable) countable chain condition (<u>c.c.c.</u>) if every collection of pair-wise disjoint uncountable sets in $B(X)$ is countable. The existence of uncountable c.c.c. sets can be demonstrated using the Continuum Hypothesis (CH). The well-known Lusin and Sierpinski sets are of this type: see [2] and [4]. Define

$$K = \{t \varepsilon S : t = t(X), \quad X \text{ c.c.c.}\}.$$

Then K is an ideal in S closed under countable sums.

<u>1.1 Proposition</u>: The partially ordered set K is a conditionally complete lattice. In fact

$$t \wedge \sup\{t_\alpha : \alpha \varepsilon A\} = \sup\{t \wedge t_\alpha : \alpha \varepsilon A\}$$

for each $t \varepsilon S$ and each collection $\{t_\alpha : \alpha \varepsilon A\}$ bounded above by an element of K. Also,

$$t \vee \inf\{t_\alpha : \alpha \varepsilon A\} = \inf\{t \vee t_\alpha : \alpha \varepsilon A\}$$

whenever t and t_α are elements of K.

<u>1.2 Proposition</u>: Let t and t_α be elements of K for $\alpha \varepsilon A$. Then

$$t + \inf\{t_\alpha : \alpha \varepsilon A\} = \inf\{t + t_\alpha : \alpha \varepsilon A\}.$$

If the family $\{t_\alpha : \alpha \varepsilon A\}$ is bounded above, we have

$$t + \sup\{t_\alpha : \alpha \varepsilon A\} = \sup\{t + t_\alpha : \alpha \varepsilon A\}.$$

<u>1.3 Proposition</u>: Let t_1 and t_2 be elements of K. Then

$t_1 + t_2 = (t_1 \vee t_2) + (t_1 \wedge t_2)$.

1.4 Example: Let $X \subseteq \mathbf{R}$ be an uncountable, measurably rigid set. Partition X into uncountable Borel sets $X = X_1 \cup X_2 \cup X_3$. Put $t_i = t(X_i)$. Then $(t_1+t_2) \vee (t_2+t_3) = t_1 + t_2 + t_3 \neq (t_1+t_2) + (t_2+t_3)$. Thus, the operations of $\vee$ and $+$ are distinct.

1.5 Example: Let $t \in S$ be rigid. Then $nt \neq mt$ for $n \neq m$ (including $n = \omega$).

1.6 Example: Let A be an infinite Borel subset of $\mathbf{R}$. Then $nt(A) = t(A)$ for all n (including $n = \omega$).

1.7 Proposition: Suppose that $s \leq t$ are elements of K. Then there is a largest element c in K such that $t = s + c$. For this c, if $0 < x \leq c$, then $s < s + x$.

Given $s \leq t$ in K, we define $c(t,s)$ to be the element c of proposition 1.7.

1.8 Proposition: Let $t_1\ t_2 \ldots$ be a sequence in S. Then

$$t_1 + t_2 + \cdots = \sup\{t_1+\cdots+t_n : n\geq 1\}.$$

§2. Covers of 0

Say that $t \in S$ is a cover of 0 if $t > 0$ and $t \geq s$ implies $s = t$ or $s = 0$. For $X \subseteq \mathbf{R}$, we see that $t = t(X)$ is a cover of 0 if and only if X is Borel-isomorphic with each of its uncountable Borel subsets. If A_1 and A_2 are uncountable Borel subsets of $\mathbf{R}$, then

$t(A_1) = t(A_2)$ (see [3]); it follows that $t(\mathbb{R}) = t(A_1)$ is a cover of 0. We show, under assumption of CH, that many such covers exist.

2.1 Proposition (CH): Let $a > 0$ be an element of S. There is some $x \in K$ such that

1) $x \wedge a = 0$

2) x is a cover 0.

2.2 Corollary (CH): There are at least c^+ covers of 0 in K, where c^+ is the cardinal successor of the continuum.

The corollary follows from the proposition *via* Fact 0.3 and an inductive argument. These covers of 0 can actually be chosen as isomorphism types of Sierpiński sets. Are there 2^c of them?

§3. Join-irreducible types

An element s in the lattice K is *join-irreducible* if $s = s_1 \vee s_2$ implies $s = s_1$ or $s = s_2$. Clearly, 0 and every cover of 0 in K is join-irreducible. Define $\sec(t) = \{s : s \leq t\}$.

3.1 Proposition: An element t of K is join-irreducible if and only if $\sec(t)$ is linearly ordered.

3.2 Proposition (CH): There is a Sierpinski set $X \subseteq [0,1]$ of outer Lebesgue measure $\lambda^*(X) = 1$ such that sets A,B in $\mathcal{B}(X)$ are Borel-isomorphic if and only if $\lambda^*(A) = \lambda^*(B)$.

Putting $t_0 = t(X)$, we see that $t_0 \in K$ is join-irreducible (proposition 3.1); in fact, we have

3.3 Corollary (CH): With t_0 as above, the lattice $\sec(t_0)$ is order-isomorphic to the linearly ordered set [0,1].

Remarkably, there is a sort of converse to this proposition as follows:

3.4 Proposition: Let t be an element of K. Then t is join-irreducible if and only if exactly one of the following obtains:

Case 1: $t = 0$.

Case 2: t is a cover of 0.

Case 3: $t > 0$ is not a cover of 0; indeed, $t \neq \omega t$. Then there is a Sierpiński set $X \subseteq [0,1]$ with $t = t(X)$ and $\lambda^*(X) = 1$ such that sets A and B in $\mathcal{B}(X)$ are Borel-isomorphic if and only if $\lambda^*(A) = \lambda^*(B)$.

Case 4: $t > 0$ is not a cover of 0, yet $t = \omega t$. Then there is a Sierpiński set $X \subseteq \mathbf{R}$ with $t = t(X)$ and $\lambda_*(\mathbf{R} \setminus X) = 0$ such that sets A and B in $\mathcal{B}(X)$ are Borel-isomorphic if and only if $\lambda^*(A) = \lambda^*(B)$.

§4. References

[1] Bhaskara Rao, K.P.S. and Rao, B.V., Borel spaces, Dissertationes Mathematicae CXC 1981.

[2] Brown, J.B. and Cox, G.V., Classical theory of totally imperfect spaces, Real Analysis Exchange 7 (1981-2) 185-232.

[3] Kuratowski, K., Topology, Volume I, Academic Press-PWN, New York-Warszawa 1966.

[4] Miller, A.W., Special subsets of the real line, Handbook of Set-Theoretic Topology, Elsevier, Amsterdam (1984) 201-234.

[5] Shortt, R.M. and van Mill, J., Automorphism groups for measurable spaces, Topology and Its Appl. (to appear).

[6] Tarski, A., Cardinal Algebras, Oxford Univ. Press, New York 1949.