Vincenzo Aversa, Dipartimento Matematico-Statistico, Via Partenope 36, 80121 Napoli, Italy

David Preiss, Department of Analysis, Charles University, Sokolovská 83, 18600 Prague 8, Czechoslovakia.

HEARTS DENSITY THEOREMS

The Lebesgue density theorem is clearly one of the most important results of real analysis. On the other hand, from the general point of view it just claims that one out of the many differentiation systems on the real line has the density property. However, since the Lebesgue density theorem (LDT) plays the key role in so many questions of real analysis, one might expect that it is canonical in some sense, e.g. that it is the only density theorem connected in a reasonable way with the algebraic structure of the reals. In the language of abstract density topologies this question was asked by L. Zajíček at Scuola di Analisi Reale, Ravello 1985 and in [Z].

Here we intend, after giving a precise formulation of the problem, to point out several other density theorems connected with the algebraic structure of the reals. This shows, surprisingly enough, that the LDT is not canonical in the sense of translation invariance nor in the sense of affine invariance.

Let us recall that in a general differentiation system (DS) (for the Lebesgue measure (denoted by $|.|$) on the real line R) to each $x \in R$ there correspond Moore-Smith sequences (families filtering to the right) of sets of finite and positive measure. (Cf. [HP].) A point $x \in R$ is said to be a density point of a measurable set $E \subset R$ (for the given DS) if

$$\lim |E \cap E_t| / |E_t| = 1$$

for every Moore-Smith sequence corresponding to x.

In an obvious way one defines the notion of the density property (DP): A DS has the DP if almost every point of every measurable set is its density point.

For example, in the Lebesgue differentiation system (LDS) to each $x \in R$ there corresponds the Moore-Smith sequence of intervals $(x-h, x+h)$ (with $h \searrow 0$). In this language the LDT says that the LDS has the DP.

For any DS Θ one of the following situations may happen:

(a) Whenever x is a density point of E for the LDS, it is also a density point of E for Θ. In this case the DP for Θ (obviously) follows from the LDT.
(b) There is some measurable set E such that some $x \in R$ is a Lebesgue density point but not Θ density point of E. If, nevertheless, Θ has the DP, we shall say that the DP for Θ does not follow from the LDT.

To illustrate this, let us consider the following examples.
(a) The DP for the DS assigning to each $x \in R$ the sequence $(x, x+1/n)$ follows from the LDT.
(b1) The DP for the DS assigning to each $x \in R$, $x \neq 0$ the sequence $(x, x+1/n)$ and to $x = 0$ the sequence $(1,n)$ does not follow from the LDT.

The example (b1) might seem to be cheating. However, the reader may construct more sofisticated examples using the following argument:
(b2) Whenever $g:R \to R^2$ is a Borel isomorphism, the DS defined as the inverse image of, say, the interval basis has the DP provided that, of course, g carries the one-dimensional Lebesgue measure to the two-dimensional Lebesgue measure.

We will not consider (b2) in detail since more concrete examples will be given later.

Motivated by (b1) and (b2), one might believe that, if a DS has the DP then this property follows from the LDT provided that the DS is defined for each $x \in R$ "in the same way". To make this notion more precise, we shall say that the DS is translationally invariant (TI) if for any Moore-Smith sequence S_i corresponding to x and for any translation t the sequence $t(S_i)$ corresponds to $t(x)$. Thus our question is whether for TIDS the DP necessarily follows from the LDT. Noting that a TIDS is given by one family of Moore-Smith sequences (in the following special cases just by one sequence), namely by those sequences corresponding to zero, we may formulate the negative answer in the following way.
(b3) The TIDS given by the sequence $(k/(k+1)!, 1/k!)$ $(k = 1, 2,\dots)$ has the DP which does not follow from the LDT.

Clearly, the DP for this system cannot follow from the LDT since zero is a Lebesgue density point of the set

$R - \bigcup_{k=1} (k/(k+1)!, 1/k!)$. Surprisingly enough, to show the DP is also quite easy:

<u>Theorem 1</u> (Second Hearts Density Theorem, cf. [P]). If I_k is a sequence of intervals converging to zero such that

$$\limsup \operatorname{diam}(I_{k+1} \cup \{0\})/|I_k| < \infty$$

then the TIDS given by I_k has the DP.

Proof. Let $S_k = I_k \cup \{0\}$ and $S_k^* = \{x \in R;\ S_k \cap \bigcup_{p \geq k} (x+S_p) \neq \emptyset\}$.

If I_k has end points $a_k < b_k$ then

$S_k^* \subset [-b_k, -a_k] \cup [a_k-b_k, b_k-a_k] \cup [a_k-C(b_k-a_k), b_k+C(b_k-a_k)]$,

where $C = \sup\{\operatorname{diam}(S_{k+1})/|S_k|\ ;\ k+1,2,\dots\}$. Hence

$$|S_k^*| \leq (C+4)(b_k-a_k) = (C+4)|S_k| .$$

It follows that the TIDS given by S_k has the halo property. Consequently it has the DP (cf. [HP], Theorem 2.2).

The next natural step is to consider affine invariant (AI) differentiation systems. They are defined in the same way as the TIDS; only "translations" are replaced by "affine bijections". Again, an AIDS is given by a single family of Moore -Smith sequences.

Our next result shows that for AIDS the situation is more complicated than for TIDS.

<u>Theorem 2.</u> If I_k is a sequence of intervals such that the AIDS given by it has the DP then

(i) the sequence I_k converges to zero, and

(ii) $\liminf |I_k| / \operatorname{diam}(I_k \cup \{0\}) > 0$.

Hence the DP for this DS follows from the LDT.

Proof. Let Θ be the AIDS given by a sequence I_k of intervals.

If $\limsup \operatorname{diam}(I_k \cup \{0\}) > 0$, one easily sees that sets of sufficiently small diameter contain no Θ density points.

Suppose that $\lim \operatorname{diam}(I_k \cup \{0\}) = 0$ and

$$\liminf |I_k| / \operatorname{diam}(I_k \cup \{0\}) = 0.$$

Then it is easy to find a subsequence k_j such that

$I_{k_j} = [a_j, b_j] \to 0$ and $|I_{k_j}| / \operatorname{diam}(I_{k_j} \cup \{0\}) \to 0$

so quickly that the set $E = R - \bigcup_j F_j$, where

$F_j = \bigcup_{n=-\infty}^{\infty} (n|a_j|/j - 2j(b_j-a_j),\ n|a_j|/j + 2j(b_j-a_j))$

has positive Lebesgue measure.

For each $x \in R$ let

$$G_{x,j} = \{r \in R;\ (x + rI_{k_j}) \subset F_j\}.$$

We claim that whenever $x \in R$, $j=1,2,\ldots$, $r \in R$, and $|r| \leq j$ then $G_{x,j} \cap (r-1/j,\ r+1/j) \neq \emptyset$. Indeed, if $s \in R$ is such that $n = j(x+sa_j)/|a_j| \in Z$, and $|j(x+sa_j)/|a_j| - j(x+ra_j)/|a_j|| < 1$, then $|s-r| < 1/j$ and, since the interval $x+sI_{k_j}$ contains $n|a_j|/j$ and has length $\leq |s||I_{k_j}| \leq 2j|I_{k_j}|$, $s \in G_{x,j}$.

Hence our claim is proved. Using it, we can see that for each $p = 1,2,\ldots$ the set $\bigcup_{j=p}^{\infty} G_{x,j}$ is a dense open subset of R. Consequently $G_x = \bigcap_{p=1}^{\infty} \bigcup_{j=p}^{\infty} G_{x,j}$ is a dense subset of R. Finally, we note that for every $x \in R$ and every $r \in G_x - \{0\}$ there are infinitely many values of k for which $(x+sI_k) \cap E = \emptyset$, which shows that Θ does not have the DP.

The above result, (b1), and (b2) seem to suggest that for AIDS the DP necessarily follows from the LDT. Again, in the language of abstract density topologies this conjecture was formulated by L. Zajíček [Z]. However, our last results disprove even this conjecture:

<u>Theorem 3.</u> There is a sequence S_k of measurable subsets of R converging to zero such that

(i) zero is a Lebesgue density point of $R - \bigcup S_k$, and

(ii) the AIDS given by S_k has the DP.

It might be worthwhile to reformulate (ii) in a more normal language:

(ii') Every measurable set E contains a subset N of measure zero such that for every $x \in E-N$ and every $r \neq 0$

$$\lim |(x+rS_k) \cap E| / |rS_k| = 1.$$

Our final result has the flavour of the "sphere theorems" of Stein and Waigner [SW] (for R^n, $n \geq 3$) and Bourgain [B] (for R^2). Using a double sequence, it can be also formulated in the language of DS.

Theorem 4. There is a continuous probability measure μ on R such that

(i) $|spt(\mu)| = 0$, and

(ii) for every Lebesgue measurable set E almost every point $x \in E$ (with respect to the Lebesgue measure) fulfils

$$\lim_{r \to 0} \mu((E-x)/r) = 1.$$

The proof of Theorems 3 and 4 will appear elsewhere. Here we just remark that it uses probabilistic methods and hence it does not give any concrete examples. For example, the following question is open: Does the probability measure associated with the Cantor set in the natural way have the property described in Theorem 4?

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