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T. Natkaniec, Department of Mathematics, Pedagogical University of Bydgoszcz, ul. Chodkiewicza 30, 85-064 Bydgoszcz, Poland.

SOME REMARKS ON CATEGORY PROJECTIONS OF PLANAR SETS

The authors of [1] provide that the measure projection of the subset A*B of R^2 is non-empty and open whenever A and B are measurable sets with positive finite Lebesgue measure /the assumption that A and B have finite measure may be omited/.

In Proposition 1 the same conclusion is proved for second category A having the property of Baire and second category B. This fact is an improvement of Theorem 2.6 of [1] /see also [2], Th. 2/.

In [5] Sierpiński constructed a second category set $S \subset R^2$ which meet every line at most in 2 points.

In Proposition 2 we improve the construction of S and give an example of a linear set A of second category for which category projections of A*A are empty. This fact is a stronger version of the Theorem of [3].

We use the notation introduced in [2]/and [1]/. Let $E \subset R^2$. By $P_m(E)$ /$R_m(E)$/ we denote the projection /the category projection/ of E in direction m.

Recall that

$$P_m(E) = \{c \in R: \quad gr(y=mx+c) \cap E \neq \emptyset\}$$

and

$$R_m(E) = \{c \in R: \quad dom[gr(y=mx+c) \cap E] \text{ is of second category}\}.$$

In this paper we assume that $m \neq \emptyset$.

Notice that $R_m(A \times B) = \{c \in R: \ (mA+c) \cap B$ is of second category$\}$.

LEMMA 1. If $A \subset R$ is a second category set then there exists an open /and non-empty/ set $G \subset R$ such that A is of second category at every point $x \in G$ and the set $A \setminus G$ is of first category.

P r o o f . Let B be the set of all $x \in R$ such that A is of first category at x and let $G = int(R \setminus B)$. The set G has the desired properties.

LEMMA 2. /a/ If $A \triangle A_1$ and $B \triangle B_1$ /the symmetric differences/ are of first category then $R_m(A \times B) = R_m(A_1 \times B_1)$.

/b/ If a set A has the property of Baire, i.e. $A = G \triangle K$, where G is an open set and K is of first category then $R_m(A \times B) = R_m(G \times B)$.

/c/ If G,H are open sets and $B \subset H$ is of second category at every point $x \in H$ then $P_m(G \times H) = P_m(G \times B) = R_m(G \times B)$.

P r o o f . The parts /a/ and /b/ are obvious.

/c/ The inclusion $R_m(G \times B) \subset P_m(G \times B) \subset P_m(G \times H)$ are clear. Let $c \in P_m(G \times H)$. Then $y = mx + c$ for some $x \in G$ and $y \in H$. Since the set $(mG+c) \cap H$ is open and non-empty, the set $(mG+c) \cap B$ is of second category and therefore $P_m(G \times H) \subset P_m(G \times B)$. If $c \in P_m(G \times B)$ then $(mG+c) \cap B$ is non-empty. Since B is of second category at every point of B, the set $(mG+c) \cap B$ is of second category. Hence $P_m(G \times B) \subset R_m(G \times B)$.

PROPOSITION 1. If either of second category sets A and B has in addition the property of Baire, then the set $R_m(A \times B)$ is open and non-empty.

P r o o f . Assume that A has the property of Baire and G is the non-empty open set of Lemma 2.b.

Let H be an open set such that: B is of second category at every point $x \in H$, $B_1 = B \cap H$ and $B \setminus B_1$ is of first category.

By Lemma 2.c we have $R_m(G \times B_1) = P_m(G \times H)$. Since $P_m(G \times H) = H - mG$ /see e.g. [1]/, it follows that the set $P_m(G \times H)$ is open and non-empty. It follows from Lemma 2.a that $R_m(A \times B) = R_m(G \times B_1) = P_m(G \times H)$.

The case when A does not have the property of Baire and B has this property is analogous.

PROPOSITION 2. There exists a second category set A such that the set $A \times A$ meets every non-horizontal and non-vertica line, except of the line $y=x$, at most in 2 points.

P r o o f . Let G_α, $\alpha < \omega_c$ be a well-ordering of all residual G_δ subsets of the line.

Choose $x_0 \in G_0$, $x_1 \in G_1$. Suppose we have chosen x_β for all $\beta < \alpha$. Put $A_\alpha = \{x_\beta : \beta < \alpha\}$.

Let $\mathcal{P}_\alpha$ denotes the family of all non-horizontal and non-vertical lines, different from the line $y=x$, which meet the set $A_\alpha \times A_\alpha$ at least in 2 points.

Put $B_\alpha = \{x: \exists p \in \mathcal{P}_\alpha \quad \exists y \in A_\alpha \ [(x,y) \in p \vee (y,x) \in p \vee (x,x) \in p]\}$,

and $C_\alpha = \{x: \exists y,z,t,w \in A_\alpha \ [(x,y), (z,x)$ and (t,w) are collinear$]\}$.

Observe that the sets B_α and C_α have cardinality less than continuum.

At level α choose $x_\alpha \in G_\alpha \setminus (B_\alpha \cup C_\alpha)$.

By letting $A = \{x_\alpha : \alpha < \omega_c\}$, it is relatively straightforward to show that A has the desired properties.

COROLLARY 1. There exists a second category set $A \subset R$ with $R_m(A \times A) = \emptyset$ for $m \notin \{0,1\}$ and $R_m(A \times A) = \{0\}$ for $m=1$.

COROLLARY 2. There exists a second category, Lebesgue measurable set $C \subset R$ with $R_m(C \times C) = \emptyset$ for $m \notin \{0,1\}$ and $R_m(C \times C) = \{0\}$ for $m=1$.

P r o o f . Let $B \subset R$ be a first category set of full measure /see e.g. [4], Corollary 1,7/.

Let A be a second category set for which the conclusion of Corollary 1 holds. Then the set $C = A \cup B$ has the desired properties.

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