Real Analysis Exchange Vol. 12 (1986-87)

Richard G. Gibson, Department of Mathematical Sciences, Columbus College, Columbus, Georgia, 31993; and Fred Roush, Department of Mathematics, Alabama State University, Montgomery, Alabama, 36101.

THE RESTRICTIONS OF A CONNECTIVITY FUNCTION ARE NICE BUT NOT THAT NICE

Let X and Y be topological spaces. A function $f:X\to Y$ is said to be a connectivity function provided that if A is a connected subset of X, then the graph of f restricted to A is a connected subset of $X\times Y$. A function $f:X\to Y$ is said to have property (s) or to be (s)- measurable provided that for each perfect set $P\subset X$ there exists a perfect set $Q\subset P$ such that the restriction $f|Q$ is continuous. Marczewski defined property (s) for sets in [8] and showed that (s)- measurable functions and the class of functions (functions with property (s)) studied by Sierpinski in [10] were the same. For further study the reader is referred to [1] and [2]. A real-valued function f defined on an interval is said to have a perfect road at the point x provided that there exists a perfect set P such that x is a bilateral point of accumulation of P and such that $f|P$ is continuous at x.

Let $I = [0,1]$. From [4] it follows that if $g:I^2\to I$ is a connectivity function, then $f = g|(I\times\{x\})$ has the following property: if $[a,b]\subset I$, then there exists a Cantor set $C\subset(a,b)$ such that $f|C$ is continuous where I is assumed to be embedded in I^2 as $I\times\{x\}$ for any $x\in I$. From [6] it follows that if $g:I^2\to I$ is a connectivity function, then $f = g|(I\times\{x\})$ has a perfect road at each point for any $x\in I$. However, there exist connectivity functions $I\to I$ that have neither of these properties [3].

The purpose of this paper is to construct a connectivity function $g:I^2 \to I$ and show that $f = g|(I \times \{x\})$ does not have property (s) for some $x \in I$. Thus the restrictions of a connectivity function $g:I^2 \to I$ are nice but not that nice.

The following construction is a variation of a construction given in [5]. For this construction we give the definition of a peripherally continuous function. A function f is said to be peripherally continuous provided that for any x and any pair of open sets U and V containing x and $f(x)$, respectively, there exists an open set W such that $x \in W \subset U$ and $f(bd(W)) \subset V$ where bd = boundary. Connectivity functions and peripherally continuous functions defined on certain spaces and in particular $I^n \to I$ are equivalent whenever $n \geq 2$,[7].

Example.

L_1: Let g be 0 on the boundary of I^2 and 1 at the center of I^2. Draw horizontal and vertical lines through the center which will divide I^2 into four small squares. Let g be linear on the edges of the small squares. The variation on the edges of these small squares ≤ 1.

L_2: Let g be 0 or 1 in a checkerboard pattern at the center of the small squares of L_1. Divide each small square of L_1 into 16 smaller squares. Let g be linear on the edges of these smaller squares and such that the variation on the edges of these squares $\leq \frac{1}{2}$. Continuing in this manner we have

L_{n+1}: Let g be 0 or 1 in a checkerboard pattern at the center of the small squares constructed in L_n. Divide each small square of L_n into $(2(n+1))^2$ smaller squares. Let g be linear on the edges of these smaller squares and such that the variation on the edges of these squares $\leq 1/(n+1)$.

By construction g is peripherally continuous on $\bigcup_{n=1}^{\infty} L_n$.

Let H_m be the set of $x \in I^2$ such that there exist squares A_0 and A_1 such that the variation on the edges of A_0 and A_1 is less than $1/m$, $x \in \mathrm{int}(A_0)$, $x \in \mathrm{int}(A_1)$, $g(\mathrm{bd}(A_0)) \subset [0,1/m)$, and $g(\mathrm{bd}(A_1)) \subset (1-1/m,1]$ where int = interior. It follows that any point of $\mathrm{int}(A_0) \cap \mathrm{int}(A_1)$ is in H_m. So H_m is open and dense in I^2. Thus $H = \bigcap_{m=1}^{\infty} H_m$ is a dense G_δ-subset of I^2. Now $I^2 - \bigcup_{n=1}^{\infty} L_n$ is a dense G_δ-subset of I^2. So $G = H \cap (I^2 - \bigcup_{n=1}^{\infty} L_n)$ is a dense G_δ-subset of I^2 on which the values of g can be chosen to be either 0 or 1. From [9] there exists an $x_0 \in I$ such that $I \times \{x_0\}$ contains a dense G_δ-subset of $G \cap (I \times \{x_0\})$. Thus it contains a Cantor set $C \subset G \cap (I \times \{x_0\})$. Define g on C to be discontinuous on every sub-Cantor set of C, taking values of 0 or 1 only.

Now let $x \in I^2 - (C \cup (\bigcup_{n=1}^{\infty} L_n))$. For each n, x is contained in the interior of a square S_n such that as $n \to \infty$, $S_n \to x$, and the variation of $S_n \to 0$. Let $x_n \in \mathrm{bd}(S_n)$. Then $x_n \to x$. Let $g(x)$ be a cluster point of $g(x_n)$. Thus $g: I^2 \to I$ is a connectivity function.

Let I be embedded in I^2 as $I \times \{x_0\}$. Then $f = g|(I \times \{x_0\})$ is a connectivity function which does not have property (s).

Remarks. The set of x_0's for which this is true is a set of the second category, [9]. Also it follows from the construction that there exist 2^c connectivity functions $I^2 \to I$, and hence there exist connectivity functions that are not of Baire class 1.

REFERENCES

1. J. B. Brown and G. V. Cox, "Classical theory of totally imperfect spaces", Real Analysis Exchange 7 (1982), 1-39.
2. ________ and K. Prikry, "Variations on Lusin's Theorem", to appear in Trans. AMS.
3. R. G. Gibson and F. Roush, "The Cantor intermediate value property", Topology Proc. 7 (1982), 55-62.
4. ____________, "Concerning the extension of connectivity functions", Topology Proc. 10 (1985), 75-82.
5. ____________, "Connectivity functions defined on I^n", Colloquium Math., (to appear).
6. ____________, "Connectivity functions with a perfect road", Real Analysis Exchange 11 no. 1 (1985-86), 260-264.
7. M. Hagan, "Equivalence of connectivity maps and peripherally continuous transformations", Proc. of A. M. S. 17 (1966), 175-177.
8. E. Marczewski (Szpilrajn), "Sur un classe de fonctions de M. Sierpinski et la classe correspondante d'ensembles", Fund. Math. 24 (1935), 17-34.
9. J. C. Oxtoby, Measure and Category, Graduate Texts in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York.
10. W. Sierpinski, "Sur un probleme de M. Ruziewicz concernant les superpositions de fonctions jouissant de la propriete de Baire", Fund. Math. 24 (1935), 12-16.

Received May 5, 1986