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Generalized Differentiation and Summability

For x real let $A_n(x) := a_n \cos nx + b_n \sin nx$ and let

$$T(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} A_n(x) \tag{1}$$

be a trigonometric series. Suppose that at every $x \in [0,2\pi)$, $T(x) = 0$. Then all a_n and b_n are zero. This is the fundamental theorem in the subject of uniqueness of trigonometric series. It was announced by Riemann in 1854 and the last detail of his proof was supplied in a letter from H.A. Schwarz to Cantor who published it in 1870.[4],[6],[7] The crucial step is this theorem.

Theorem R. If F is continuous and

$$RF(x) := \lim_{h\to 0} \frac{F(x-h) - 2F(x) + F(x+h)}{h^2}$$

is zero everywhere, then F is a line.

Theorem R is immediate from a lemma.

Lemma R. If F is continuous and $RF \geq 0$ everywhere then F is convex. (See [7],vol.I, p.23, Theorem 10.7.)

Theorem R has only one known proof, namely via Lemma R. To extend

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Theorem R to higher dimensional settings it could be useful to have another proof.[1],[2],[3] The search for a new proof might be aided by first proving analogs of theorem R for other generalized second derivatives of the form $\lim_{h\to 0^+} h^{-2}\sum_{i=1}^{3} a_i f(x+b_i h)$, where $\Sigma\, a_i = \Sigma\, a_i b_i = 0$ and $\Sigma\, a_i b_i^2 = 2$. These are quickly seen to classify into three types: type I has $b_1 < 0 = b_2 < b_3$, type II has all of the b_i of the same sign, and type III has $b_1 < 0 < b_2 < b_3$ or $b_1 < b_2 < 0 < b_3$. Interesting new phenomena arise only from type III. The case of $b_1 = -1$, $b_2 = 1$, and $b_3 = 2$ is typical of this type. This motivates the definition

$$A^+F := \lim_{h\to 0^+} \frac{(2/3)F(x+2h) - F(x+h) + (1/3)F(x-h)}{h^2}.$$

Since the continuous non-convex

$$u(x) := \left\{ \begin{array}{ll} 0 & x < 0 \\ -\left[x^{\log_2(3/2)}\right] & x \geq 0 \end{array} \right\}$$

enjoys $A^+u \geq 0$ for all x, the analog of Lemma R is false here. (See [2], pp.19-20. If one allows $h\to 0^-$ as well, then the situation of continuous non-convex f with $Af \geq 0$ everywhere does not arise.) For this reason the truth of the following conjecture would very likely increase our knowledge in the area of uniqueness of trigonometric series.

Conjecture A. If F is continuous and A^+F is zero everywhere, then F is linear.[2]

In an attempt to prove this, I was led to a related

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[2] On August 10, 1986 I was informed that M. Laczkovich and P. Humke have found a proof for this conjecture.

summability result. Let $F(x)$ be a continuous function with Fourier series (1). Differentiate this series twice termwise, thereby forming the distributional second derivatives $F'' := \Sigma -n^2 A_n(x)$. An elementary computation shows

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = \sum -n^2 A_n(x) \left[\frac{\sin \frac{nh}{2}}{\frac{nh}{2}}\right]^2 .$$

By definition $RF(x) := \lim_{h\to 0} \left[\frac{F(x+h) - 2F(x) + F(x-h)}{h^2}\right]$ and by definition the series F'' is summable R to s if and only if

$s = \lim_{h\to 0} \left[\sum -n^2 A_n(x) \left[\frac{\sin \frac{nh}{2}}{\frac{nh}{2}}\right]^2\right]$. Thus theorem R can be restated by saying that a continuous function whose distributional second derivative is summable R everywhere to 0 is linear. Similarly the derivative A^+ corresponds to a method of summability. Define a series $\Sigma\, a_n$ to be summable A^+ to s if

$$\lim_{h\to 0^+} \sum_{n=-\infty}^{\infty} a_n \gamma(inh) = s \quad \text{where}$$

$$\gamma(t) := \frac{(2/3)e^{2t} - e^t + (1/3)e^{-t}}{t^2}.$$

(Note that the series (1) must be expanded in complex exponential form before applying the multiplier γ. The infinite sum is defined as the limit of the symmetric partial sums.) As with the Riemann situation, we have $A^+F(x)$ exists if and only if the twice formally differentiated Fourier series of F is summable A^+.

There is a theorem of Kuttner [5] that summability R implies Abel summability and a theorem of Verblunsky ([7], vol.I, Theorem 7.4) stating that if a trigonometric series is Abel summable to 0 everywhere and has coefficients $o(n)$, then all coefficients are 0. (Recall that $\Sigma\, c_n$ is Abel summable to s if $\lim_{r\to 1^-} \Sigma\, c_n r^n = s$.) I had hoped to show Conjecture A by first showing summability A^+ implies Abel summability, then controlling the coefficients, and finally applying Verblunsky's theorem. The following result extinguished that hope.

<u>Theorem.</u> Summability A^+ and Abel summability are not comparable.

<u>Proof.</u> First we show that summability A^+ does not imply Abel summability. The function $u(x)$ above, restricted to $[-\pi,\pi)$ and then extended periodically, thus has u'', its distributional second derivative, summable A^+ to 0 at 0. (See [2], p.19.) However u'' is not Abel summable at 0. To show this we must prove that

$$\lim_{r\to 1^-} \Sigma\, -n^2 A_n(0) r^n = \lim_{r\to 1^-} \Sigma\, n^2(-a_n) r^n$$

does not exist. Letting $\alpha := \log_2(3/2) \approx .58$ we have

$$-a_n = \mathcal{R}e\, \left\{ \frac{1}{\pi} \int_0^\pi x^\alpha e^{-inx} dx \right\} .$$

Throughout the following calculation we will repeatedly discard purely imaginary terms.

Substitute $t := nx$ and then integrate by parts, differentiating t^α and integrating e^{-it}, to get

$$-a_n = \mathcal{R}e\, \left\{ \frac{-i\alpha n^{-1-\alpha}}{\pi} \int_0^{n\pi} t^{\alpha-1} e^{-it} dt \right\} .$$

Writing $\int_0^{n\pi} = \int_0^{\infty} - \int_{n\pi}^{\infty}$ and using $\int_0^{\infty} t^{\alpha-1}e^{-it}dt = \Gamma(\alpha)e^{-i\frac{\pi}{2}\alpha}$ ([7], vol.I, p. 69) yields

$$-a_n = \mathcal{R}e\left\{\frac{-i\alpha n^{-1-\alpha}}{\pi}\left[\Gamma(\alpha)(-i \sin \tfrac{\pi}{2}\alpha) - \int_{n\pi}^{\infty} t^{\alpha-1}e^{-it}dt\right]\right\} .$$

Next integrate by parts three more times, always differentiating the power of t and integrating the e^{-it} factor. We obtain

$$-a_n = \frac{-\Gamma(\alpha+1)\sin \frac{\pi}{2}\alpha}{\pi} n^{-1-\alpha} + \frac{\alpha}{\pi^{2-\alpha}} (-1)^n n^{-2} + c_n n^{-4} ,$$

where the constants c_n are uniformly bounded. Hence

$$\Sigma\, n^2(-a_n)r^n = \frac{-\Gamma(\alpha+1)\sin \frac{\pi}{2}\alpha}{\pi}\Sigma\, n^{1-\alpha} r^n + \frac{\alpha}{\pi^{2-\alpha}}\Sigma\, (-1)^n r^n + \Sigma\, c_n n^{-2} r^n .$$

As $r \to 1^-$, the third term tends to the convergent series $\Sigma\, c_n n^{-2}$, $\sum_{n=1}^{\infty} (-1)^n r^n = \frac{-r}{1+r}$ which tends to $-\frac{1}{2}$, but $\lim_{r\to 1^-} \Sigma\, n^{1-\alpha} r^n = +\infty$.

Conversely, let v(x) be the periodic extension of $(\operatorname{sgn} x)\sqrt{|x|}$ restricted to $[-\pi,\pi)$. A simple calculation shows that $A^+v(0)$ does not exist, so that v" is not summable A^+ at 0. However, since v is odd, v" is a pure sine series, so that $A_n(0) = 0$ for all n. Hence v" is summable Abel to 0 at 0.

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