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ON A THEOREM OF SHIZUO KAKUTANI

In a paper from 1942 ([1]) Shizuo Kakutani proved the following statement: "let $f(P)$ be a real-valued continuous function defined on a two-sphere S^2. Then there exists a triple of points P_1, P_2, $P_3 \in S^2$ perpendicular to one another, such that $f(P_1) = f(P_2) = f(P_3)$". In the same paper he asks whether the property is still valid when replacing the two-sphere S^2 by a (n-1)-sphere S^{n-1} and the 3 points P_1, P_2, P_3 by n points $P_1,\dots,P_n$ ($n \geq 4$). This problem is still open.

In this paper we prove a plane version of Kakutani's theorem; namely, we replace S^2 by $\bar{\mathbb{C}}$ and the three perpendicular points by the vertices of an equilateral triangle: if f is a continuous mapping of $\bar{\mathbb{C}}$ into $\mathbb{R}$, there exists a triple of points z_1, z_2, $z_3 \in \mathbb{C}$ such that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| > 0$ and $f(z_1) = f(z_2) = f(z_3)$.

The main result we use in proving the nontrivial case of this statement is the following theorem: if $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, then $\partial\Omega$ contains the vertices of an equilateral triangle. This statement is known if $\partial\Omega$ is a Jordan arc, but an example we give at the end of the proof of the main theorem shows that this weaker form is not sufficient.

We start by proving the result in the strong form.

<u>THEOREM 1</u>: If $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, then $\partial\Omega$ contains the vertices of an equilateral triangle.

<u>PROOF</u>: We need three lemmas.

<u>LEMMA 1</u>: If $\Omega \subset \mathbb{C}$ is a bounded, simply connected domain, there exist some points x_1, $x_2 \in \partial\Omega$ and $x_3 \in \Omega$ such that $|x_1 - x_2| = |x_2 - x_3| = |x_3 - x_1| > 0$.

PROOF: Let $x_0 \in \Omega$ and let $D = \{z \in \mathbb{C} : |z - x_0| < r\}$ be an open disc such that $D \subset \Omega$ and $\partial D \cap \partial\Omega \neq \emptyset$. (Such a disc exists, since $\partial\Omega$ is closed.) Choose $x_1 \in \partial D \cap \partial\Omega$ and consider the rotations $\mathcal{R}, \mathcal{R}' : \mathbb{C} \to \mathbb{C}$, $\mathcal{R}(z) = x_1 + (z - x_1)(\cos \pi/3 + i \sin \pi/3)$ and $R'(z) = x_1 + (z - x_1)(\cos 5\pi/3 + i \sin 5\pi/3)$. Denote $\mathcal{R}(D)$ by D_1, $\mathcal{R}'(D)$ by D_2, $\mathcal{R}(D_1)$ and D_3 and $\mathcal{R}'(D_2)$ by D_4.

If $D_1 \cap \partial\Omega \neq \emptyset$ or $D_2 \cap \partial\Omega \neq \emptyset$, the lemma obviously holds.

Otherwise, i.e. if $D_1 \cap \partial\Omega = D_2 \cap \partial\Omega = \emptyset$, Ω connected forces $D_1, D_2 \subset \Omega$, and it will be enough to prove that $D_3 \cap \partial\Omega \neq \emptyset$ or $D_4 \cap \partial\Omega \neq \emptyset$.

This statement holds since $D \cup D_1 \cup D_2 \cup D_3 \cup D_4$ covers $D^* = \{z \in \mathbb{C} : 0 < |z - x_1| < r\}$ and $\partial\Omega$ is infinite and connected, so the lemma is proved.

LEMMA 2: Let $\Omega \subset \mathbb{C}$ be bounded, let $x_1 \in \partial\Omega$, and let $\mathcal{R}$ be any of the rotations $z \mapsto x_1 + (z - x_1)(\cos \pi/3 + i \sin \pi/3)$ and $z \mapsto x_1 + (z - x_1)(\cos 5\pi/3 + i \sin 5\pi/3)$. Then there exist some points x_4, x_5 such that $x_4 \in \partial\Omega$, $x_5 = \mathcal{R}(x_4)$ and $x_5 \in C\Omega$ (sign "C" meaning taking of the complement with respect to $\mathbb{C}$).

PROOF: Choose x_4 to be a point of $\bar{\Omega}$ such that $|x_1 - x_4| = \sup\{|z - x_1| : z \in \bar{\Omega}\}$. Since the supremum is attained on $\partial\Omega$ only, we must have $x_4 \in \partial\Omega$ and $x_5 = \mathcal{R}(x_4) \in C\Omega$. Q.E.D.

LEMMA 3: If $\Omega_1, \Omega_2 \subset \mathbb{C}$ are bounded, simply connected domains such that $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\Omega_1 \cap \text{Int}(C\Omega_2) \neq \emptyset$ and $\Omega_2 \cap \text{Int}(C\Omega_1) \neq \emptyset$, then $\text{Card}(\partial\Omega_1 \cap \partial\Omega_2) \geq 2$.

PROOF: Since obviously $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$, we must have $\text{Card}(\partial\Omega_1 \cap \partial\Omega_2) \geq 1$.

Suppose $\text{Card}(\partial\Omega_1 \cap \partial\Omega_2) = 1$ and denote $\partial\Omega_1 \cap \partial\Omega_2$ by p. Also denote by K a connected component of $C(\Omega_1 \cup \Omega_2 \cup \{p\})$ such that $K \cap \partial\Omega_1 \neq \emptyset$ and $K \cap \partial\Omega_2 \neq \emptyset$ (It is easily seen that such a component exists.) and choose $a \in \Omega_1 \cap \Omega_2$ and $b \in K$.

We claim first that $\overline{\Omega}_1 \cap \partial\Omega_2$ does not separate a and b (i.e. a and b lie in the same connected component of $C(\overline{\Omega}_1 \cap \partial\Omega_2)$).

To see this, denote $C\Omega_1 \cap \partial\Omega_2$ by L; since $K \cap \partial\Omega_2 \neq \emptyset$ and $K \cap (\overline{\Omega}_1 \cap \partial\Omega_2) = \emptyset$ (for $\overline{\Omega}_1 \cap \partial\Omega_2) \cup (\overline{\Omega}_2 \cap \partial\Omega_1) = (\Omega_1 \cap \partial\Omega_2) \cup (\Omega_2 \cap \partial\Omega_1) \cup (\partial\Omega_1 \cap \partial\Omega_2) \subset (\Omega_1 \cup \Omega_2 \cup \{p\}) \subset CK$), we have $K \cap L \neq \emptyset$, and since $L \cup \Omega_2$ is connected (for $\Omega_2 \subset (\Omega_2 \cup L) \subset \overline{\Omega}_2$ and Ω_2 is connected), $K \cup \Omega_2 \cup L$ is also connected. Since $a, b \in (K \cup \Omega_2 \cup L) \subset C(\overline{\Omega}_1 \cap \partial\Omega_2)$, it follows that $\overline{\Omega}_1 \cap \partial\Omega_2$ doesn't separate a and b indeed.

Now since an analogous statement is valid for $\overline{\Omega}_2 \cap \partial\Omega_1$ and since $\overline{\Omega}_1 \cap \partial\Omega_2$ and $\overline{\Omega}_2 \cap \partial\Omega_1$ are closed and $(\overline{\Omega}_1 \cap \partial\Omega_2) \cap (\overline{\Omega}_2 \cap \partial\Omega_1) = \{p\}$ is connected, it follows, by Janiszewski's first theorem (See [2], p. 284.) that $(\overline{\Omega}_1 \cap \partial\Omega_2) \cup (\overline{\Omega}_2 \cap \partial\Omega_1)$ doesn't separate a and b. But since $\partial(\Omega_1 \cap \Omega_2) \subset (\overline{\Omega}_1 \cap \partial\Omega_2) \cup (\overline{\Omega}_2 \cap \partial\Omega_1)$, this forces $\partial(\Omega_1 \cap \Omega_2)$ not to separate a and b. Contradiction.

So $\mathrm{Card}(\partial\Omega_1 \cap \partial\Omega_2) \neq 1$, and the lemma is proved.

Back to the proof of the theorem, let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain, let x_1, x_2, x_3 be the three points given by lemma 1, and take z_1 to be x_1. Denote by $\mathcal{R}$ the rotation of C around x_1 which transforms x_2 into x_3 and set $\Omega' = \mathcal{R}(\Omega)$. Let also x_4, x_5 be the points given by lemma 2 applied to Ω.

If $x_5 \in \partial\Omega$, we may take $z_2 = x_4$ and $z_3 = x_5$.

If $x_5 \notin \partial\Omega$, by lemmas 1 and 2 we have $\Omega \cap \Omega' \neq \emptyset$, $\Omega \cap \mathrm{Int}(C\,\Omega') \neq \emptyset$ and $\Omega' \cap \mathrm{Int}(C\Omega) \neq \emptyset$, so by lemma 3 we may choose $z_2 \in (\partial\Omega \cap \partial\Omega')\setminus\{z_1\}$. If we set now $z_3 = \mathcal{R}^{-1}(z_2)$, the proof is complete.

We may now state the main theorem.

<u>THEOREM 2</u>: Let f be a continuous mapping of $\overline{\mathbb{C}}$ into $\mathbb{R}$. Then there exists a triple of points $z_1, z_2, z_3 \in \mathbb{C}$ such that $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| > 0$ and $f(z_1) = f(z_2) = f(z_3)$.

<u>PROOF</u>: If f is constant, the theorem follows obviously, so let f be nonconstant. Since $\overline{\mathbb{C}}$ is compact and connected, $f(\overline{\mathbb{C}})$ will be also compact and connected, so we may choose $\alpha \in \mathrm{Int}(f(\overline{\mathbb{C}})\setminus\{f(\infty)\})$. Let $f^{-1}(\alpha) = \{z \in \overline{\mathbb{C}} : f(z) = \alpha\}$ by X and suppose $\mathrm{Int}\,X = \emptyset$. (If $\mathrm{Int}\,X \neq \emptyset$, the theorem clearly holds.) Also denote CX by Y.

We obviously have $X = \partial Y$, and Y has at least two connected components, precisely one of them being unbounded. Let K be a bounded component of Y, let L be the unbounded component of CK, and denote CL by Ω.

Since $\partial\Omega \subset \partial K \subset \partial Y = X$, to complete the proof it will be enough to show that Ω is simply connected.

Since $C\Omega = L$ is connected, we have to show only that Ω is connected.

Suppose Ω disconnected. We may find then two nonempty disjoint sets Ω_1 and Ω_2, closed in Ω, such that $\Omega_1 \cup \Omega_2 = \Omega$.

We claim that $K \cap \Omega_1 \neq \emptyset$.

Suppose $K \cap \Omega_1 = \emptyset$. Then we have $L \cup \Omega_1 \subset CK$; since L is connected and Ω_1 is simultaneously open and closed in $\Omega = CL$, $L \cup \Omega_1$ will also be connected; so $L \cap \Omega$ lies in a connected component of CK, and this component must be precisely L (since $L \cap (L \cup \Omega_1) \neq \emptyset$), contradiction!

So we must have $K \cap \Omega_1 \neq \emptyset$.

For the same reasons we also have $K \cap \Omega_2 \neq \emptyset$, and since obviously $K \subset \Omega_1 \cup \Omega_2$, we have $K = (K \cap \Omega_1) \cup (K \cap \Omega_2)$, and $(K \cap \Omega_1)$, $(K \cap \Omega_2)$ are disjoint and closed in K. This is absurd since K is connected. Hence Ω is connected indeed.

By theorem 1, our statement is proved.

Let us make some remarks.

First, let us see that the weak form of theorem 1 is not sufficient for proving the main theorem.

Construct f as follows: consider the sets

$$X_1 = \{z = x + i\,y \in \mathbb{C} : x \in (0,1],\ y = \sin \tfrac{1}{x}\}\ ,$$

$$X_2 = \{z = x + i\,y \in \mathbb{C} : x = 1,\ y \in [-2,\ \sin 1)\}\ ,$$

$$X_3 = \{z = x + i\,y \in \mathbb{C} : x \in [0,1),\ y = -2\} \quad \text{and}$$

$$X_4 = \{z = x + i\,y \in \mathbb{C} : x = 0,\ y \in [-2,1]\}\ .$$

Denote $X_1 \cup X_2 \cup X_3 \cup X_4$ by X', and denote by Ω' the set $\{z = x + i y \in \mathbb{C} : x \in (0,1),\ y \in (-2, \sin \frac{1}{x})\}$; define a function $g : \mathbb{C} \to \mathbb{R}$, $g(z) = 0$ if $z \in X'$, $g(z) = \sup\{|z - z'| : z' \in X'\}$ if $z \in \Omega'$ and $g(z) = -\sup\{|z - z'| : z' \in X'\}$ if $z \in \mathbb{C} \setminus (X' \cup \Omega')$. Now set $f(z) = \exp(g(z))$ if $z \in \mathbb{C}$ and $f(\infty) = 0$.

This is clearly a continuous function, and if we choose $\alpha = 1$, we have $X' = f^{-1}(1)$, $\Omega = \Omega'$, $X' = \partial\Omega$, and $\partial\Omega$ is not a Jordan arc. This justifies our strengthening of the theorem.

Remark also that what we have actually proved is that several triangles with required properties exist - in the nontrivial case one for each range value, except for endpoints and $f(\infty)$.

Finally, we may state also an analogous of the generalized Kakutani theorem. Namely, we may ask whether, if $\bar{\mathbb{R}}^{n-1}$ is the one-point compactification of $\mathbb{R}^{n-1}$ $(n \geq 4)$ and f is a real-valued continuous mapping of $\bar{\mathbb{R}}^{n-1}$, we may find a regular simplex with vertices $P_1,\dots,P_n \in \mathbb{R}^{n-1}$ such that $f(P_1) = \cdots = f(P_n)$? Or, equivalently, given a bounded, simply connected domain $\Omega \subset \mathbb{R}^{n-1}$, does it follow that $\partial\Omega$ contains the vertices of a regular $(n-1)$ - dimensional simplex?

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[1] Kakutani, S., A proof that there exists a circumscribing cube around any bounded closed convex set in $\mathbb{R}^3$, Ann. Math., (2), 43, 739-741, (1942);

[2] Kuratowski, K., Introduction to the set theory and topology, PWN - Polish Scientific Publishers, Warszawa, 2-nd edition 1972.

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