INROADS

Real Analysis Exchange Vol. 12 (1986-87)

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DERIVATIVES OF TYPE 1

1. Functions of type k.

The main obstacle in attempts to characterize the class $b\Delta$ of bounded derivatives stems from the fact that this class is not closed under outside composition with continuous functions. For example, the following holds (See [1], page 138.):

If $f \in b\Delta$ and $f^2 \in b\Delta$, then f is approximately continuous.

From this result one easily sees that every subclass of $b\Delta$ admitting a topological characterization or a characterization in terms of associated sets is contained in the class bA of bounded approximately continuous functions.

There are some bounded derivatives whose properties change after an outside composition with a continuous function in a rather drastic way. Nevertheless, there are also some approximately discontinuous derivatives which behave well even after such a compostion.

Example (See [3], Chapter II, §1, no. 6, exercise 7.):

$$\text{Put}\quad f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

If $\varphi:\mathbb{R} \to \mathbb{R}$ is a continuous function, then the function $\varphi \circ f$ may fail to be a derivative, but it is easy to see that

$$\lim_{s\to 0} s^{-1} \int_0^s \varphi(f(t))dt = \pi^{-1} \int_{(-1,1)} \varphi(y)d(\arc \sin y).$$

Thus the function

$$g(x) = \begin{cases} \varphi(f(x)) & \text{if } x \neq 0 \\ \pi^{-1} \int_{(-1,1)} \varphi(y)d(\arc \sin y) & \text{if } x = 0 \end{cases}$$

is a derivative.

In a similar way we can construct a bounded derivative h which is approximately discontinuous at any point of a given perfect set of measure zero, such that for any continuous function φ and any $x \in \mathbb{R}$

$\lim_{s\to 0} s^{-1} \int_x^{x+s} \varphi(h(t))dt$ exists. Again, there is a function $g \in b\Delta$ such that $\varphi \circ h = g$ a.e.

From these examples it seems to be clear that the class of those bounded functions f such that for any continuous function φ one may find $g \in b\Delta$ such that $\varphi \circ f = g$ a.e., is a rather interesting (and topologically characterizable! - see theorem 3) class of "almost derivatives". (Note that "almost derivative" means "equivalent to a derivative" and not "being a generalized derivative".)

The above examples suggest also some possibility of classification of bounded derivatives. To explain it, denote by $\mathcal{C}$ the space of all continuous functions on $\mathbb{R}$ endowed with the topology of locally uniform convergence, by $\mathcal{C}^*$ the dual of $\mathcal{C}$ with the weak* topology (i.e. the space of signed measures with compact support) and by $\mathfrak{M}$ the set of all nonnegative measures in $\mathcal{C}^*$ with total mass one. Recall also that for $\mu \in \mathfrak{M}$ the barycenter $r(\mu)$ is defined by $r(\mu) = \int_{\mathbb{R}} t \, d\mu(t)$.

Let $f:\mathbb{R} \to \mathbb{R}$ be a bounded measurable (with respect to the Lebesgue measure λ) function. We note that f is a derivative of its indefinite integral at a point $x \in \mathbb{R}$ iff $\lim_{I\to x} (\lambda(I))^{-1} \int_I f(t) \, d\lambda(t) = f(x)$. (Here $I \to x$ means: I is an interval, $x \in I$ and $\lambda(I) \to 0$.) Rewriting

$$(*) \qquad (\lambda(I))^{-1} \int_I f(t)d\lambda(t) = \int_{\mathbb{R}} t \, df[(\lambda \llcorner I)/\lambda(I)](t)$$

where $(\lambda \llcorner I)(E) = \lambda(I \cap E)$ and where the image $f[\mu]$ of a measure μ is defined by $f[\mu](E) = \mu(f^{-1}(E))$ for all Borel sets E), we easily see that the question of whether f is a derivative of its indefinite integral at x or not depends on the set $M_x(f)$ of all limit points of the measures $f[(\lambda \llcorner I)/\lambda(I)]$ as $I \to x$. More precisely, we define $M_x(f) =$

$\bigcap_{\varepsilon>0} \mathcal{Cl}\, \{f[(\lambda \llcorner I)/\lambda(I)];\ I$ is an interval, $x \in I$ and $\lambda(I) < \varepsilon\}$. The above discussion together with the compactness of the sets $\{\mu \in \mathfrak{M};$ support $\mu \subset [-K,K]\}$ easily gives the assertions (a),(b) of the following proposition.

Proposition 1: Let $f:\mathbb{R} \to \mathbb{R}$ be a bounded measurable function and let $x \in \mathbb{R}$. Then

(a) $M_x(f)$ is a nonempty compact subset of M.

(b) f is a derivative of its indefinite integral at x iff $r(\mu) = f(x)$ for each $\mu \in M_x(f)$.

(c) f is approximately continuous at x iff $M_x(f) = \{\varepsilon_{f(x)}\}$, where ε_y denotes the Dirac measure concentrated at y.

(d) If $\varphi:\mathbb{R} \to \mathbb{R}$ is a continuous function, then $M_x(\varphi \circ f) = \{\varphi[\mu],\ \mu \in M_x(f)\}$.

(e) The following statements are equivalent:

(i) For any homeomorphism $\varphi:\mathbb{R} \to \mathbb{R}$ the indefinite integral of $\varphi \circ f$ is differentiable at x.

(ii) For any continuous function $\varphi:\mathbb{R} \to \mathbb{R}$ the indefinite integral of $\varphi \circ f$ is differentiable at x.

(iii) $M_x(f)$ contains exactly one measure.

The assertion (d) follows from (*) and from the formula $\int_{\mathbb{R}} \varphi \, df[\mu] = \int_{\mathbb{R}} \varphi \circ f \, d\mu$ which holds whenever φ is a Borel function and one of the integrals exists. The statement (e) follows from (d) and (b).

One can easily see that (b) can be replaced by the stronger statement:

$$\{r(\mu);\ \mu \in M_x(f)\} = [\underline{D}(\textstyle\int f,x),\ \overline{D}(\textstyle\int f,x)].$$

The above proposition leads us to the following classification of functions and derivatives:

Definition 1: A bounded function of the first class is said to be of type k if, for any $x \in \mathbb{R}$, the dimension of the linear span of $M_x(f)$ is at most k.

The set of all functions of type k we shall denote by $\mathcal{T}_k$. If $f \in \mathcal{T}_k \cap b\Delta$, we say that f is a derivative of type k. It is not difficult to see that the classes $\mathcal{T}_k$ are closed under uniform convergence and under outside composition with continuous functions. Hence they are characterizable in terms of pseudouniformities. (See theorem 3.) However, for $k \geq 2$ the families $\mathcal{T}_k$ are rather far from derivatives and even the derivatives of type 2 seem to have no nice properties.

Before passing to the description of our results concerning the functions of type 1, let us note the following corollary of proposition 1:

<u>Proposition 2</u>: Whenever $\mathcal{T}$ is a family of bounded functions of the first class such that

(i) $\varphi \circ f \in \mathcal{T}$ for any $f \in \mathcal{T}$ and any homeomorphism $\varphi:\mathbb{R} \to \mathbb{R}$

(ii) for each $f \in \mathcal{T}$ there is a derivative g such that $f = g$ a.e.,

then $\mathcal{T} \subset \mathcal{T}_1$.

Hence one can interpret the class $\mathcal{T}_1$ as a maximal "characterizable" subfamily of "almost derivatives".

2. <u>The functions of type 1</u>.

Let $f:\mathbb{R} \to \mathbb{R}$ be a function of type 1 and let $x \in \mathbb{R}$. Then, by definition, $M_x(f)$ contains exactly one measure and we shall denote this measure by μ_x^f. From proposition 1 it follows that

$$\lim_{h \to 0} h^{-1} \int_{(x,x+h)} \varphi(f(t))\, dt = \int_R \varphi \, d\, \mu_x^f$$

for any continuous function $\varphi:\mathbb{R} \to \mathbb{R}$. It is not difficult to show the following:

If $S \subset \mathbb{R}$ is a Borel set such that $\mu_x^f(\partial S) = 0$, then the density of $f^{-1}(S)$ at x exits and $d(x,f^{-1}(S)) = \mu_x^f(S)$.

For a function $f:\mathbb{R} \to \mathbb{R}$ we shall denote by A_f the set of the points at which f is approximately continuous. It is not difficult to show that if $f \in \mathcal{T}_1$, then A_f is of type G_δ.

The main result concerning the functions of type 1 is the following:

Theorem 1. Let $f: \mathbb{R} \to \mathbb{R}$ be a function of type 1, $S \subset \mathbb{R}$ a G_δ-set of measure zero containing $\mathbb{R} - A_f$, ε a positive number and $Y = \{y_1, \ldots, y_m\} \subset \mathbb{R}$ a finite set such that $\operatorname{dist}(f(x), Y) < \varepsilon/4$ for any $x \in \mathbb{R}$. Then there exists a function h of type 1 such that $A_h \supset A_f$, $|f-h| < \varepsilon$, and, for any $x \in S$, $\mu_x^h = \sum_{i=1}^{m} a_i(x)\, \varepsilon_{y_i}$.

The following theorem gives a characterization of level sets of derivatives of type 1:

Theorem 2. For a set $E \subset \mathbb{R}$ the following conditions are equivalent:

(i) $E = \{x, f(x) > 0\}$ for some $f \in \mathcal{J}_1 \cap b\Delta$

(ii) There exist a sequence $\{F_n\}$ of closed sets, a sequence $\{A_n\}$ of measurable sets and a sequence of positive numbers $\{\eta_n\}$ such that, for each n, $F_n \subset A_n \subset E = \bigcup_{n=1}^{\infty} F_n$, the density of A_n at each of its points exists, and $d(x, A_n) > \eta_n$ for any $x \in F_n$.

(iii) $E = \{x, f(x) > 0\}$ for some $f \in \mathcal{J}_1 \cap b\Delta$ and $f \geq 0$.

Remarks. 1. It seems to be interesting to compare the condition (ii) from the preceding theorem with an equivalent definition of the Zahorski class M_4:

Lemma. A set $E \subset \mathbb{R}$ belongs to the class M_4 iff there exist a sequence of closed sets $\{F_n\}$ and a sequence of positive numbers $\{\eta_n\}$ such that $E = \bigcup_{n=1}^{\infty} F_n$ and for each $x \in F_n$ there exists a set $B \subset E$ having a density at x such that $d(x, B) > \eta_n$.

The lemma follows easily from [2].

2. It is easy to construct an M_4-set which does not satisfy the condition (ii) from theorem 2.

Finally, let us briefly consider the problem of characterizing the functions of type 1 in terms of pseudouniformities. We refer to [4] for basic definitions pertaining to uniform spaces.

Definition 2: A pseudouniformity on a set X is a system μ of covers of X satisfying the following conditions:

(i) $\mathcal{U} \in \mu$, $\mathcal{U} < \mathcal{V} \Rightarrow \mathcal{V} \in \mu$

(ii) $\mathcal{U} \in \mu \Rightarrow$ there exists $\mathcal{V} \in \mu$ such that $\mathcal{V} < {}^*\mathcal{U}$.

A pseudouniform space is a set together with a pseudouniformity on it.

In pseudouniform spaces we can define the notions of the basis of pseudouniformity and uniformly continuous functions in an obvious way.

The following theorem shows which systems of functions can be described as functions uniformly continuous with respect to some pseudouniformity.

Theorem 3. For a system $\mathcal{F}$ of real functions on a set X the following conditions are equivalent:

(i) There is a pseudouniformity μ on X such that $f \in \mathcal{F}$ iff f is μ-uniformly continuous.

(ii) The system F is closed under uniform convergence and satisfies the following condition:

If $f \in \mathcal{F}$ and $\varphi: f(X) \to \mathbb{R}$ is a function uniformly continuous on $f(X)$, then $\varphi \circ f \in \mathcal{F}$.

In particular, if all functions in $\mathcal{F}$ are bounded, then $\mathcal{F}$ satisfies (i) iff it is closed under uniform convergence and under outside composition with continuous functions.

From the remark after definition 1 it follows that there is a pseudouniformity μ on $\mathbb{R}$ such that μ-uniformly continuous functions are exactly the functions of type 1. We are able to give an explicit description of a basis of such a pseudouniformity.

References.

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Received July 8, 1983