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Josef Tkadlec, MFF UK, Sokolovská 83, 186 75 Praha 8, Czechoslovakia

Construction of a finite Borel measure with σ-porous sets as null sets

It is well-known and easy to see that each finite Borel measure on the read line whose null sets contain all sets that are of Lebesgue measure zero as well as of the first category is necessarily absolutely continuous with respect to Lebesgue measure. We show that in this statement one cannot replace the notion of the first categroy sets by the more restrictive notion of σ-porous sets (introduced by Dolženko [1]). Namely, we construct a finite Borel measure μ on the real line such that each σ-porous set is a μ-null set and μ is not absolutely continuous with respect to Lebesgue measure. In the construction we use a special case of a general construction of perfect, non-σ-porous sets given in [2], where also other differences between the class of σ-porous sets and the class of sets of the first category and of Lebesgue measure zero are presented.

For every bounded, open (closed) interval I and for every positive real number c we denote by $c*I$ the open (closed) interval with the same center as I and with length $|c*I| = c.|I|$.

Lemma 1. Let S be a σ-porous subset of the real line and let $c > 1$. Then there is a sequence $\{S_n\}_{n=1}^{\infty}$ of porous sets such that $S = \bigcup_{n=1}^{\infty} S_n$ with the following property for every positive integer n. For every $x \in S_n$ and for every $t > 0$ there exists an open interval $I \subset (x - t, x + t)\setminus S_n$ such that $x \in c*I$.

Proof. It easily follows from [3], theorem 4.5.

Lemma 2. Let μ be a finite Borel measure on a subset S of the real line and let the following conditions hold.

(1) There is $d > 1$ such that $\sum\mu(d*I) < \infty$; the summation being over the set of all bounded intervals I contiguous to $\bar{S}$.

(2) There are $c > 1$, $C > 0$ and $\delta > 0$ such that $\mu(c*I) \le C\mu(I)$ for every interval I with $|I| < \delta$ and with center in S.

(3) Countable sets are σ-null sets.

Then $\mu(P) = 0$ for every σ-porous set P.

Proof. By induction we obtain from condition (2) that $\mu(c^n*I) \le C^n\mu(I)$ for every positive integer n and for every interval I with $|I| < \delta\cdot c^{-n+1}$ and with center in S. Hence we may suppose $c \ge \frac{3d+1}{2}\cdot\frac{d+1}{d-1}$. We may also suppose $P \subset S \subset (0,1)$. According to lemma 1 we need only prove that $\mu(P) = 0$ for every porous set P such that for every $x \in P$ and for every $t > 0$ there is an open interval $I \subset (x-t, x+t)\backslash P$ with $x \in \frac{d+1}{2}*I$. Denote by $\{I_n^1\}_{n=1}^{\infty}$ ($\{I_n^2\}_{n=1}^{\infty}$) the sequence of all components of $(0,1)\backslash\bar{P}$ for which $\frac{d+1}{2d}*I_m \cap S$ is empty (nonempty). Then

$$P \subset \limsup_{n\to\infty}(\frac{d+1}{2}*I_n^1) \cup \limsup_{n\to\infty}(\frac{d+1}{2}*I_n^2) \cup \bigcup_{n=1}^{\infty} \text{bdry } I_n^1 \cup \bigcup_{n=1}^{\infty} \text{bdry } I_n^2 .$$

For every interval I_n^1 except at most two there is a bounded interval I_n continguous to $\bar{S}$ such that $I_n^1 \subset \frac{2d}{d+1}*I_n$, $\mu(\frac{d+1}{2}*I_n^1) \le \mu(d*I_n)$. Hence, according to condition (1),

$$\mu(\limsup_{n\to\infty}(\frac{d+1}{2}*I_n^1)) = 0 .$$

For every interval I_n^2 except at most a finite number there is an interval $J_n \subset I_n^2$ with center in S and satisfying $\frac{d-1}{2d}|I_n^2| \le |J_n| < \delta$. Hence $I_n^2 \subset \frac{3d+1}{d-1}*J_n$ and, according to condition (2), $\mu(\frac{d+1}{2}*I_n^2) \le \mu(\frac{3d+1}{2}\cdot\frac{d+1}{d-1}*J_n) \le C\mu(J_n)$. Because the intervals J_n are pairwise disjoint and μ is a finite measure,

$$\mu(\limsup_{n\to\infty}(\frac{d+1}{2}*I_n^2)) = 0 .$$

From condition (3) we obtain $\mu(P) = 0$.

__Construction__. Let

(a) $\{k_n\}_{n=1}^{\infty}$ be the nondecreasing sequence of positive integers containing each positive integer m exactly 2^m times, and

(b) $P_n(R)$ be the set of points, which decompose the interval R into 2^{k_n+2} closed subintervals of equal length (card $P_n(R) = 2^{k_n+2} - 1$) for every bounded closed interval R and for every positive integer n. We denote by $\mathcal{R}_n(R)$ the system of all such subintervals except those containing the center of the interval R.

By induction we define systems $\mathcal{R}_n$ of closed intervals such that $\mathcal{R}_0 = \{[0,1]\}$ and $\mathcal{R}_n = \cup\{\mathcal{R}_n(R);\ R \in \mathcal{R}_{n-1}\}$ for every positive integer n. We let $S = \bigcap_{n=0}^{\infty} \cup\{R;\ R \in \mathcal{R}_n\}$ and by induction define the mapping $\tau : \bigcup_{n=0}^{\infty} \mathcal{R}_n \to [0,1]$ such that $\tau([0,1]) = 1$ and such that for every positive integer n and for every interval $R \in \mathcal{R}_n$ and for $R' \in \mathcal{R}_{n-1}$ with $R \subset R'$

$$\tau(R) = \begin{cases} 2^{-2k_n-1}\tau(R') , & R \subset 2^{-k_n} {*} R' , \\ 3\cdot 2^{-k_n-k-2}\tau(R') , & \operatorname{Int} R \subset 2^{-k+1} {*} R' \setminus 2^{-k} {*} R' , \\ & k \in \{1,\dots,k_n\} . \end{cases}$$

Since $\sum_{\{R\in\mathcal{R}_n;\, R\subset R'\}} \tau(R) = \tau(R')$ for every $R' \in \mathcal{R}_{n-1}$, there is a Borel measure μ such that $\operatorname{supp} \mu = S$ and $\mu(I) = \tau(I)$ for every $I \in \bigcup_{n=0}^{\infty} \mathcal{R}_n$.

Remark. According to a proposition in [2] the set S is non-σ-porous. (The corresponding system $\mathcal{D}_n(R) = P_n(R) \cup \{2^{-k_n-1} {*} R\}$ fulfils both conditions (C1) and (C2) for every positive integer n and for every closed interval R because condition (C2) holds whenever $\mathcal{D}_n(R)$ contains at most one nondegenerated interval.)

Theorem. There exists a finite Borel measure on the real line that is zero on every σ-porous set and that is not absolutely continuous with respect to Lebesgue measure.

Proof. We constructed the finite Borel measure μ and a set S such that $\mu(S) = 1$, supp $\mu = S$ and such that countable sets are μ-null sets. The Lebesgue measure of the set S,

$$|S| = \prod_{n=1}^{\infty} (1 - 2^{-k_{n-1}})$$

is zero because $\sum_{n=1}^{\infty} 2^{-k_{n-1}} = \sum_{m=1}^{\infty} 2^{m} \cdot 2^{-m-1} = \infty$. We need only to prove that conditions (1) and (2) of lemma 2 hold. Note that

$$\sum_{n=1}^{\infty} \sum_{R \in \mathcal{R}_{n-1}} \mu(2_*(2^{-k_{n-1}}{}_*\mathrm{Int}\ R)) = \sum_{n=1}^{\infty} \sum_{R \in \mathcal{R}_{n-1}} 2^{-2k_n}\mu(R) =$$

$$= \sum_{n=1}^{\infty} 2^{-2k_n} = \sum_{m=1}^{\infty} 2^{m} \cdot 2^{-2m} = 1 < \infty .$$

Suppose J is an interval with center $x \in S$ such that $|J| < 2$. Let n be the smallest positive integer such that there are intervals $R' \in \mathcal{R}_n$, $R \in \mathcal{R}_{n-1}$ such that $x \in R' \subset J \cap R$. Let $Q = J \cap \bigcup_{k=0}^{k_n+1} \mathrm{bdry}\ 2^{-k}{}_*R$. We distinguish two cases.

1) card $Q \leq 1$. Then $J \cap 2^{-k_{n-1}}{}_*R = \emptyset$. The interval J contains K intervals from $\mathcal{R}_n$ and the set $S \cap 2_*J$ is contained in the union of $2K + 5$ intervals from $\mathcal{R}_n$ whose μ-measure is at most twice that of $S \cap 2_*J$. Hence

$$\frac{\mu(2_*J)}{\mu(J)} \leq \frac{2(2K + 5)}{K} \leq 14 .$$

2) card $Q \geq 2$. Then let k be the smallest positive integer in $\{1,\dots,k_n + 1\}$ such that $J \cap \mathrm{bdry}\ 2^{-k+1}{}_*R \neq \emptyset$. Hence

$$\mu(J) \geq \frac{1}{2}\, \mu(2^{-k+1}{}_*R \setminus 2^{-k}{}_*R)$$

Since $2_*J \subset 2^{k+3} R$,

$$\mu(2_*J) \leq \mu(2^{-k+3}{}_*R \setminus 2^{-k+2}{}_*R) + \mu(2^{-k+2}{}_*R \setminus 2^{-k+1}{}_*R) +$$

$$+ \mu(2^{-k+1}{}_*R \setminus 2^{-k}{}_*R) + \mu(2^{-k}{}_*R) \leq$$

$$\leq (4 \cdot 4 + 2 \cdot 2 + 1 + 1)\mu(2^{-k+1}{}_{*}{*r} \setminus 2^{-k}{}_*R)$$

Hence

$$\frac{\mu(2_*J)}{\mu(J)} \leq 44$$

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References

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