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POROSITY, $\mathcal{I}$-DENSITY TOPOLOGY AND ABSTRACT DENSITY TOPOLOGIES

Introduction.

The present article contains proofs of some results presented in my lecture on Scuola di Analisi Reale, Ravello 1985.

W. Wilczyński [13] defined the $\mathcal{I}$-density topology on R which is in a sense a category analogue of the density topology on R. The properties of the $\mathcal{I}$-density topology and its generalization to R^n were investigated in several articles (cf. [14]).

The $\mathcal{I}$-density topology was defined by W. Wilczyński as a topology determined by a special "lower density in the category sense". Topologies which are determined by an arbitrary "lower density in the category sense" (abstract category density topologies) are investigated in [6] simultaneously with the usual abstract density topologies (defined on measure spaces, cf. [12]) from an abstract point of view. In the first part of the article we state some basic results on abstract density topologies from [6] and describe a general, simple construction of abstract category density topologies. For example, to the a.e.-topology and r-topology (defined by R. J. O'Malley in [7]) there corresponds by this construction abstract category density topologies a^* and r^*.

The original definition of the $\mathcal{I}$-density topology uses the algebraic structure of R but it is possible to give a definition using topological notions and the notion of porosity only. This enables us to define in the second part of the article a generalization of the $\mathcal{I}$-density topology in an arbitrary metric space (p^*-topology). We prove several theorems concerning the p^*-topology. In particular, we answer a question from [1] which concerns the $\mathcal{I}$-density topology.

Since there exist several variants of the notion of porosity, we obtain definitions of new abstract category density topologies which are very similar to the $\mathcal{I}$-density topology. The definitions of these topologies and a discussion of some questions which arise naturally in the presented general setting are contained in the third part.

1. Abstract category density topologies.

Let Σ be a σ-algebra of subsets of a set X and let $\mathcal{N} \subset \Sigma$ be a σ-ideal. In the following we shall suppose that for any $A \subset X$ there exists a "measurable cover" H_A such that $A \subset H_A$, $H_A \in \Sigma$ and $H_A \setminus P \in \mathcal{N}$ whenever $A \subset P \in \Sigma$. We know only two interesting examples of such triples $(X,\Sigma,\mathcal{N})$:

I. (Measure case). (X,Σ,μ) is a measure space with a complete, σ-finite measure and $\mathcal{N}$ is the system of all μ-null sets.

II. (Category case). X is a topological space, Σ is the system of all subsets of X which have the Baire property and $\mathcal{N}$ is the system of all first category sets. It is easy to prove that in this case we can put

$$H_A = A \cup \{x \in X \,;\; A \cap U_x \text{ is a second category set for any neighbourhood } U_x \text{ of } x\}.$$

In the sequel we shall write $A \sim B$ if $(A \setminus B) \cup (B \setminus A) \in \mathcal{N}$. The interior, closure and boundary of a set M with respect to a topology τ are denoted by $\mathrm{int}_\tau M$, $\mathrm{Cl}_\tau M$ and $\partial_\tau M$.

Now we shall state three results from [6].

Theorem A. Let $L : \Sigma \to \Sigma$ have the following properties:

(i) $L(A) \sim A$,

(ii) $A \sim B \Rightarrow L(A) = L(B)$,

(iii) $L(\emptyset) = \emptyset$, $L(X) = X$,

(iv) $L(A \cap B) = L(A) \cap L(B)$.

Then $\{A \in \Sigma;\ A \subset L(A)\} = \{L(B) \setminus N \,;\, B \in \Sigma,\ N \in \mathcal{N}\}$, and this system forms a topology τ_L on X.

Any operator $L : \Sigma \to \Sigma$ with the properties (i) - (iv) is called a lower density on $(X,\Sigma,\mathcal{N})$ and τ_L is called the topology induced by the lower density L. A topology τ on X is said to be an abstract density topology on $(X,\Sigma,\mathfrak{N})$ if it is induced by a lower density on $(X,\Sigma,\mathfrak{N})$. In the "Category case" an abstract density topology on $(X,\Sigma,\mathfrak{N})$ is called an (abstract) category density topology on the topological space X. The following theorems give useful characterizations of abstract density topologies.

<u>**Theorem B**</u>. A toplogy τ on X is an abstract density topology on $(X,\Sigma,\mathfrak{N})$ iff the following conditions hold:

(a) $A \in \mathfrak{N} \Longleftrightarrow A$ is τ-nowhere dense and τ-closed,

(b) $A \in \Sigma \Longleftrightarrow A$ has the τ-Baire property.

<u>**Theorem C**</u>. A topology τ on X is an abstract density topology on $(X,\Sigma,\mathfrak{N})$ iff the following conditions hold:

(a) $A \in \mathfrak{N} \Rightarrow A$ is τ-closed,

(b) $A \in \Sigma \Rightarrow A \setminus \mathrm{int}_\tau A \in \mathfrak{N}$,

(c) $G \neq \emptyset$ and G is τ-open $\Rightarrow G \in \Sigma \setminus \mathfrak{N}$.

The simplest and the most important example of an abstract density topology in the "Measure case" is the ordinary density topology on the real line.

Let (P,ρ) be a topological space. Using the well-known Kuratowski theorem which asserts that a set $N \subset P$ is of the first category whenever it is of the first category at all its points, it is easy to prove that the system

$$\{G\setminus N \; ; \; G \text{ is } \rho\text{-open and } N \text{ is a } \rho\text{-first category set}\}$$

forms a topology (See, for example [8], [4] and [6].) which will be labelled ρ^*. Theorem C easily implies that ρ^* is a category density topology on (P,ρ) iff (P,ρ) is a Baire space (i.e., any nonempty open subset of P is a second category set). In this case ρ^* is obviously the coarsest category density topology on (P,ρ) which is finer than ρ. If (R,e) is the Euclidean line, the topology e^* is the simplest category density topology on (R,e). A more interesting example of a category density topology on (R,e) is the $\mathfrak{I}$-density topology.

We shall need the following simple theorem which was proved in [4] in the case when (P,ρ) is a T_1-space which is ρ^*-dense in itself and in [6] in the full generality. We shall essentially reproduce the proof from [6], p. 27.

Theorem D. Let (P,ρ) be a Baire space and let f be a real function on P. Then f is ρ^*-continuous if and only if it is ρ-continuous.

Proof. At first we shall show that for any $M \subset P$ there exists a ρ-open set G_M such that $\operatorname{int}_{\rho^*} M \subset G_M \subset \mathcal{C}_{\rho^*}M$. In fact, $\operatorname{int}_{\rho^*}M = H\setminus N$, where H is a ρ-open set and $N \subset H$ is a ρ-first category set. Since (P,ρ) is a Baire space, we easily see that $H \subset \mathcal{C}_{\rho^*}M$ and therefore we can put $G_M = H$. Now suppose that f is ρ^*-continuous. Then for any $a \in R$ we have

$$\{x ; f(x) > a\} = \bigcup_{n=1}^{\infty} G_{M_n} \qquad \text{where} \qquad M_n = \{x ; f(x) > a+n^{-1}\}$$

and therefore $\{x ; f(x) > a\}$ is ρ-open. Similarly we obtain that $\{x : f(x) < a\}$ is ρ-open and thus f is ρ-continuous.

In the sequel it will be useful to use the following terminology introduced by A.R. Todd [11].

Definition. Let τ_1 and τ_2 be topologies on a set X. We shall say that τ_1 and τ_2 are S-related if for any set $A \subset X$, $\operatorname{int}_{\tau_1} A \neq \emptyset$ iff $\operatorname{int}_{\tau_2} A \neq \emptyset$.

We shall need the following simple lemma. (See [11] and [6].)

Lemma 1. Let τ_1 and τ_2 be S-related topologies on a set X. Then for these topologies the notions of dense sets, nowhere dense sets, first category sets and sets with the Baire property coincide. Moreover, (X,τ_1) is a Baire space iff (X,τ_2) is Baire space.

An immediate consequence of Lemma 1 and Theorem B is the following fact.

Proposition 1. Let τ_1 and τ_2 be S-related topologies on X. Then a topology τ on X is a category density topology on (X,τ_1) iff it is a category density topology on (X,τ_2).

This proposition and Lemma 1 imply the following theorem which describes a simple general construction of category density topologies.

Theorem 1. Let (P,ρ) be a Baire topological space and let ω be a topology on P which is S-related to ρ. Then the topology ω^* is a category density topology on (P,ρ) and

$$\omega^* = \{G \setminus N \ ; \ G \text{ is } \omega\text{-open}, \ N \text{ is a } \rho\text{-first category set}\}.$$

Let a and r be the a.e.-topology and r-topology on R, which were defined by R. J. O'Malley in [7]. Recall that $G \subset R$ is a-open iff it is open in the density topology and $G \setminus \text{int}\, G$ is a Lebesgue null set. The r-topology has a basis of r-open sets which consists of all sets which are open in the density topology and are simultaneously G_δ and F_σ. Since both a and r are S-related to the Euclidean topology on R (See [7] or [6].), we obtain as a consequence of Theorem 1 the following corollary.

Proposition 2. The topologies a^* and r^* are category density topologies on R and $G \subset R$ is a^*-open (r^*-open, respectively) iff it is of the form $G = H \setminus N$ where H is a-open (r-open, respectively) and N is a first category set.

2. Porosity topologies.

In this part (P,ρ) will be an arbitrary metric space. Topological notions concerning ρ will be written without index (prefix) ρ. For example, the boundary of a set $M \subset P$ is denoted by ∂M. The open ball with center $x \in P$ and radius $r > 0$ is denoted by $U(x,r)$. Let $M \subset P$, $x \in P$, $R > 0$. Then we denote the supremum of the set of all $r > 0$ for which there exists $y \in P$ such that $U(y,r) \subset U(x,R) \setminus M$ by $\gamma(x,R,M)$. If

$$\limsup_{R \to 0+} \gamma(x,R,M)R^{-1} > 0,$$

we say that M is porous at x. We shall need the following obvious fact.

Lemma 2. If x is an isolated point of P, then M is porous at x iff $x \notin M$. If x is not an isolated point of P, then M is porous at x iff there exist $c > 0$ and sequences of balls $U(x,R_n)$, $U(y_n,r_n)$ such that $R_n \searrow 0$, $r_n/R_n > c$, $x \notin U(y_n,r_n)$ and $U(y_n,r_n) \subset U(x,R_n)\setminus M$.

It is easy to see that M is porous at x iff $\mathcal{C}M$ is. If x is not an isolated point of P and M is porous at x, then clearly x is a point of accumulation of $P\setminus M$.

Definition. We say that $E \subset P$ is superporous at $x \in P$ if $E \cup F$ is porous at x whenever F is porous at x. A set $G \subset P$ is said to be p-open (porosity open) if $P\setminus G$ is superporous at any point of G.

It is easy to see that E is superporous at x iff $\mathcal{C}E$ is superporous at x. The system of all sets which are superporous at x obviously forms an ideal. Therefore the system of all p-open sets forms a topology p, which will also be called the p-topology or the porosity topology. Obviously p is finer than the ρ-topology. It is easy to see that a point $x \in P$ is ρ-isolated iff it is p-isolated.

Proposition 3. Let $V \subset P$ and $x \in V$. Then the following conditions are equivalent:

(i) V is a p-neighborhood of x,

(ii) $\operatorname{int} V \cup \{x\}$ is a p-neighborhood of x,

(iii) $P\setminus V$ is superporous at x.

Proof. To prove (i) => (iii) suppose that V is a p-neighborhood of x and $\tilde{V} \subset V$ is a p-open neighborhood of x. By the definition of the p-topology $P\setminus\tilde{V}$ is superporous at x and therefore also $P\setminus V$ is superporous at x. To prove (iii) => (ii) suppose that $P\setminus V$ is superporous at x. Then also $\mathcal{C}(P\setminus V) = P\setminus \operatorname{int} V$ is superporous at x. Consequently $T := P\setminus(\operatorname{int} V \cup \{x\})$ is superporous at x. Since T is clearly superporous at all points of $\operatorname{int} V$, we obtain that $\operatorname{int} V \cup \{x\}$ is a p-open neighborhood of x. The implication (ii) => (i) is obvious.

Corollary. The porosity topology p is S-related to the ρ-topology.

__Proposition 4.__ A set $G \subset P$ is p-open iff there is an open set H and $Z \subset \partial H$ such that $G = H \cup Z$ and $P \setminus H$ is superporous at every point of Z.

__Proof.__ If $G \subset P$ is p-open, we put $H = \mathrm{int}_\rho G$ and $Z = G \setminus H$. Let $z \in Z$. Then z is not ρ-isolated and consequently by Lemma 2 $\{z\}$ is superporous at z. By Proposition 3, $H \cup \{z\}$ is a p-neighborhood of z and consequently $P \setminus (H \cup \{z\})$ is superporous at z. Therefore $P \setminus H = (P \setminus (H \cup \{z\})) \cup \{z\}$ is superporous at z as well. Clearly z is a point of accumulation of H and therefore $z \in \partial H$. The opposite implication is obvious.

__Definition.__ A subset of P is said to be superporous if it is superporous at all its points.

Proposition 3 implies that $A \subset P$ is superporous iff A is p-discrete and contains no isolated points of P.

Proposition 4 immediately implies the following fact.

__Proposition 5.__ If $A \subset P$ is p-open, then $A \setminus \mathrm{int}\, A$ is superporous.

__Definition.__ The topology p^* will be called the p^*-topology or the *-porosity topology.

By the corollary of Proposition 3 and by Theorem 1 we immediately obtain the following important fact.

__Theorem 2.__ If (P,ρ) is a Baire space, then the p^*-topology is a category density topology on P, and $G \subset P$ is p^*-open iff $G = H \setminus N$, where H is p-open and N is a first category set.

The following immediate consequence of Proposition 4 describes the structure of p^*-open sets.

__Proposition 6.__ A set $W \subset P$ is p^*-open iff there exist an open set H, $Z \subset \partial H$ and a first category set $N \subset H$ such that $W = (H \setminus N) \cup Z$ and $P \setminus H$

is superporous at any point of Z. In particular, any p^*-open set has the Baire property.

The following simple fact follows easily from Theorem C, Theorem 2, Proposition 6 and Lemma 2.

<u>Proposition 7</u>. The following conditions are equivalent:

(i) P is a Baire space,

(ii) any p^*-isolated point is isolated,

(iii) p^* is a category density topology on (P,ρ).

The following characterization of p-interior points is useful for applications.

<u>Proposition 8</u>. A set $V \subset P$ is a p-neighborhood of a point $x \in V$ iff the following condition (C) is satisfied.

(C) For any $u > 0$ there exist $d > 0$ and $v > 0$ such that whenever $U(y,r) \subset H(x,R)$ are balls for which $x \notin U(y,r)$, $R < d$ and $r/R > u$, there exists a ball $U(z,a) \subset U(y,r) \cap V$ such that $a/r > v$.

<u>Proof</u>. We can suppose that x is not an isolated point of P, the opposite case being trivial. Suppose that C is satisfied. By Proposition 3 it is sufficient to prove that $P\setminus V$ is superporous at x. Let a set $F \subset P$ which is porous at x be given. By Lemma 2 there exist $c > 0$ and sequences of balls $U(y_n,r_n)$, $U(x,R_n)$ such that $R_n \searrow 0$, $U(y_n,r_n) \subset U(x,R_n)\setminus F$, $x \notin U(y_n,r_n)$ and $r_n/R_n > c$. Find $d > 0$ and $v > 0$ which correspond to $u = c$ by (C). Let $R_{n_0} < d$. Then for any $n \geq n_0$ there exists a ball $U(z_n,a_n) \subset U(y_n,r_n) \cap V$ such that $a_n/r_n > v$. Since $U(z_n,a_n) \subset U(x,R_n)$, $a_n/R_n > c\cdot v$ and $U(z_n,a_n) \cap ((P\setminus A) \cup F) = \emptyset$, we obtain that $(P\setminus V) \cup F$ is porous at x.

To prove the opposite implication, suppose that $P\setminus V$ is superporous at x and (C) does not hold. Then there exist $u > 0$ and sequences of balls $U(y_n r_n)$, $U(x,R_n)$ such that $U(y_n,r_n) \subset U(x,R_n)$, $R_n < 1/n$, $r_n/R_n > u$, $x \notin U(y_n,r_n)$ and

(1) there is no ball $U(z_n,a_n) \subset U(y_n,r_n) \cap V$ for which $a_n/r_n > 1/n$.

Put $A := P \setminus \bigcup_{n=1}^{\infty} U(y_n, r_n/2)$. Since A is porous at x, we have that $A \cup (P \setminus V)$ is also porous at x. Consequently by Lemma 2 there exists $c > 0$ and sequences of balls $U(t_n, s_n) \subset U(x, S_n)$ such that $S_n \searrow 0$, $x \notin U(t_n, s_n)$, $s_n/S_n > c$ and $U(t_n, s_n) \subset P \setminus (A \cup (P \setminus V)) = V \cap \bigcup_{n=1}^{\infty} U(y_n, r_n/2)$. Find $n_o > 2$ such that $1/n_o < c/2$. Since $\rho(x, \bigcup_{n=1}^{n_o} U(y_n, r_n/2)) > 0$, there exist k and $n > n_o$ for which $t_k \in U(y_n, r_n/2)$. Since $\rho(x, t_k) \geq r_n/2$, we have $S_k > r_m/2$ and consequently $s_k > c \cdot r_n/2$. If we put $z_n = t_k$ and $a_n = \min(r_n/2, s_k)$, we have $U(z_n, a_n) \subset U(y_n, r_n) \cap V$ and $a_n/r_n \geq \min(1/2, c/2) > 1/n_o > 1/n$ which contradicts (1).

__Note.__ Using Proposition 8 and the characterization of $\mathcal{I}$-dispersion points given by E. Lazarow [5] (See [14], Theorem 44.) it is not difficult to prove that if (P,ρ) is the real line $\mathbb{R}$, then the p^*-topology coincides with the $\mathcal{I}$-density topology. Nevertheless, our "porosity definition" was given under the influence of some proofs from [2] and [3] independent of [5] and [14]. Another equivalent definition of the $\mathcal{I}$-density topology will be given in a subsequent article.

One of the most interesting facts about the $\mathcal{I}$-density topology is the theorem ([2], cf. [14]) which asserts that any real function which is continuous with respect to the $\mathcal{I}$-density topology is a Baire one function. We shall prove a slightly more general theorem for the p^*-topology, using a general theorem from [6]. We shall use the notion of the "essential radius condition" from [6] which in the case $P = \mathbb{R}$ almost coincides with Thomson's "intersection condition" (See [9] or [10].) for local systems.

__Definition.__ A topology τ on a metric space (P,ρ) is said to satisfy the essential radius condition if for each $x \in P$ and each τ-neighborhood U of x there is an "essential radius" $r(x,U) > 0$ such that

$$\rho(x,y) \leq \min(r(x,U_x), r(y,U_y)) \Rightarrow U_x \cap U_y \neq \emptyset$$

for every τ-neighborhoods U_x, U_y of x,y, respectively.

We shall use the next theorem which follows immediately from results of [6] (pp. 64,66).

Theorem E. Let (P,ρ) be a metric space, τ be a topology on P which satisfies the essential radius condition (w.r.t. ρ) and $f: P \to \mathbb{R}$ be a function which is τ-continuous at any point of a set $C \subset P$. Then $f|_C$ is a Baire one function (on the metric space (C,ρ)).

Note. Thomson's Lemma 2.8. from [9] (cf. [10], p. 74) implies Theorem E in the special case $C = P = \mathbb{R}$.

Theorem 3. If (P,ρ) is a Baire space, then the p^*-topology satisfies the essential radius condition.

Proof. If $x \in P$ and V^* is a p^*-neighborhood of x, then we shall determine an "essential radius" $r(x,V^*)$ in the following way. Choose a p-neighborhood V of x such that $V \setminus V^*$ is a first category set and by the condition (C) from Proposition 8, corresponding to V, x and $u = 1/3$ choose the corresponding $d = d_1(x,V) > 0$ and $v = v_1(x,V) > 0$. Further with $u = v_1(x,V)$ choose the corresponding $d = d_2(x,V)$ and $v = v_2(x,V)$ and put $r(x,V^*) = (1/3)\min(d_1(x,V), \; d_2(x,V))$. Now suppose that V^*_x is a p^*-neighborhood of x, V^*_y is a p^*-neighborhood of y and $\rho(x,y) \leq \min(r(x,V^*_x), \; r(y,V_y))$. We can suppose without loss of generality that $v_1(x,V_x) \geq v_1(y,V_y)$. Consider the balls $U(y,\rho(x,y)) \subset U(x,2\rho(x,y))$. Since $\rho(x,y)/2\rho(x,y) > 1/3$, $x \notin U(y,\rho(x,y))$ and $2\rho(x,y) < d_1(x,V_x)$, we obtain that there exists a ball $U(z,p) \subset U(y,\rho(x,y)) \cap V_x$ such that $p/\rho(x,y) > v_1(x,V_x) \geq v_1(y,V_y)$. If $y \in U(z,p)$, then we obtain from Proposition 3 that there exists an open set $\emptyset \neq H \subset V_x \cap V_y$. If $y \notin U(z,p)$, then observe that $U(z,p) \subset U(y,\rho(x,y))$, $p/\rho(x,y) > v_1(y,V_y)$ and $\rho(x,y) < d_2(y,V_y)$. Consequently there exists a ball $U(t,q) \subset U(z,p) \cap V_y$ with $q/p > v_2(y,V_y)$. In this case we also obtain an open set $\emptyset \neq H = U(t,q) \subset V_x \cap V_y$. Since P is a Baire space, we have $V^*_x \cap V^*_y H \neq \emptyset$. and the proof is complete.

As a consequence of Theorem 3 and Theorem E we obtain the following result.

Theorem 4. Let (P,ρ) be a Baire space and let $f: P \to \mathbb{R}$ be a function which is p^*-continuous at any point of a set $C \subset P$. Then $f|_C$ is a Baire one function (on the metric space (C,ρ)).

In the rest of this part we shall investigate relationships between p^*-continuity and continuity of real functions. The following result follows immediately from Theorem D.

Proposition 9. Let P be a Baire space and let f be a real function on P. Then f is p^*-continuous on P iff it is p-continuous on P.

Theorem 5. Let P be a Baire space and let f be a p^*-continuous function. Then the set $D(f)$ of all points of discontinuity of f is a countable union of closed superporous sets.

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a basis of open sets in $\mathbb{R}$. Obviously $D(f) = \bigcup_{n=1}^{\infty} (f^{-1}(B_n)\setminus \operatorname{int} f^{-1}(B_n))$. By Theorem 4 f is a Baire one function and therefore $f^{-1}(B_n)\setminus \operatorname{int} f^{-1}(B_n)$ is an F_σ-set for any n. By Proposition 9 $f^{-1}(B_n)$ is p-open and consequently $f^{-1}(B_n)\setminus \operatorname{int} f^{-1}(B_n)$ is superporous for any n by Proposition 5. Now it suffices to observe that any subset of a superporous set is superporous.

The following theorem gives an answer to query c) of [1], p. 79. The idea of the construction is the same as that of the proof of Theorem 5 from [1].

Theorem 6. Let $D \subset R$. Then there exists a p^*-continuous function f such that $D = D(f)$ iff D is a countable union of closed superporous sets.

Proof. Let $D = \bigcup_{n=1}^{\infty} A_n$ where all A_n are closed superporous sets. We can suppose that any A_n is either a perfect set or a singleton. Suppose that n is fixed, A_n is a perfect set and $\{(a_n^k, b_n^k)\}_{k=1}^{\infty}$ are all bounded intervals contiguous to A_n. Denote by (c_n^k, d_n^k) the interval concentric with (a_n^k, b_n^k) for which $b_n^k - a_n^k = 2k\,(d_n^k - c_n^k)$. Now choose a function f_n with the following properties:

(a) $0 \leq f_n \leq 3^{-n}$ and f_n is continuous on $R\backslash A_n$,

(b) $f_n(x) = 0$ for $x \in R \setminus \bigcup_{k=1}^{\infty} (c_n^k, d_n^k)$,

(c) $f_n((a_n^k + b_n^k)/2) = 3^{-n}$ for any k.

It is easy to prove that $A_n \cup \bigcup_{k=1}^{\infty} (c_n^k, d_n^k)$ is superporous at any point of A_n. This implies that

(d) f_n is p-continuous.

Obviously

(e) $\operatorname{osc}(f_n, x) = 3^{-n}$ for any point $x \in A_n$.

If A_n is a singleton, then it is not difficult to construct a function f_n which has the properties (a), (d), (e). Now it suffices to put $f = \sum_{n=1}^{\infty} f_n$.

3. Additional remarks.

If we replace in the definition of the porosity topology and the *-porosity topology the notion of porosity by the notion of (g)-porosity, we obtain definitions of new topologies: (g)-porosity topology and *-(g)-porosity topology. We say ([15]) that a set $M \subset (P, \rho)$ is (g)-porous at x if $\limsup_{R\to 0+} g(\gamma(x,R,M)) \cdot R^{-1} > 0$. Similarly we can define the strong porosity topology and the *-strong porosity topology which correspond to the notion of strong porosity. We say (cf. [16]) that a set $M \subset P$ is strongly porous at if $\limsup_{R\to 0+} \gamma(x,R,M)R^{-1} \geq 1/2$. Strong porosity was considered in [15] under the name $(x,1/2)$-porosity. Of course, it is possible to define other topologies which correspond to other porosity notions (e.g. <H>-porosity from [15]). All such defined "*-topologies" have similar properties; in particular, they are category density topologies.

An interesting question is in which sense the ordinary density topology on $\mathbb{R}$ is a "canonical" abstract density topology on $\mathbb{R}$. Of course, it is possible to answer that it is canonical because it has the simplest and the most symmetrical definition and has interesting applications. It seems to me that there may exist a "more mathematical" answer which shows that the ordinary density topology is canonical since it and only it has come simple properties. I conjectured that the ordinary density topology on $\mathbb{R}$ is the coarsest topology among all (measure) abstract density topologies on $\mathbb{R}$ which are translation invariant and finer than the Euclidean topology. D. Preiss in his lecture in Ravello (1985) proved the so-called Hearts density theorem which implies that my conjecture was false. In fact, the Hearts density theorem implies that whenever τ is a translation invariant abstract density topology on $\mathbb{R}$ finer than the Euclidean topology, there exists a topology $\tilde{\tau}$ which has the same properties and is strictly coarser than τ. It is still possible that the above conjecture is true if we replace "translation invariant" by "invariant with respect to any affine bijection".

A similar question arises with respect to the $\mathcal{I}$-density topology. It corresponds in the following sense to the ordinary density topology on $\mathbb{R}$. The original Wilczyński definition of the $\mathcal{I}$-density topology is a definition which depends on an ideal of sets $\mathcal{I}$. It $\mathcal{I}$ is the system of all first category sets, then the corresponding topology is the $\mathcal{I}$-density topology. If $\mathcal{I}$ is the system of all Lebesgue null sets, then the corresponding topology is the ordinary density topology on $\mathbb{R}$. It would be interesting to find some properties of the $\mathcal{I}$-density topology which show that it is a "canonical" category density topology on $\mathbb{R}$ or that it corresponds in some sense to the ordinary density topology on $\mathbb{R}$.

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