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$D^{\#}$ Derivation Basis and the Lebesgue-Stieltjes Integral

This paper answers a question posed by Thomson [7, p.377] on the relation between the $D^{\#}$ derivation basis and the Lebesgue-Stieltjes integral. The question is stated as follows: "The basis $D^{\#}$ can be used to characterize the Lebesgue integral. The corresponding Stieltjes integral $D^{\#}\text{-}\int f(x)dg(x)$ seems not to have been investigated, apart from several remarks in McShane." The remark in McShane, while vague, asserts that if the $D^{\#}$ Stieltjes integral exists for continuous functions f with respect to a function g, then g must be of bounded variation [2, p.40]. It will be shown that the $D^{\#}\text{-}\int f(x)dg(x)$ is the Lebesgue-Stieltjes integral for functions g of bounded variation.

The following definitions and notation will be needed. They are taken from Thomson [7, pp.87, 92, 101, 108, 116, 117, 125, 137, 157, 165].

<u>Definition</u> 1.1. The <u>sharp derivation basis</u>. For a positive function δ on R,

$\beta^{\#}_{\delta} = \{(I,x)$: I is an interval in R, $x \in R$, and

$I \subset (x-\delta(x),\ x+\delta(x))\}$ and

$D^{\#} = \{\beta^{\#}_{\delta}$: δ is a positive function on R}.

<u>Definition</u> 1.2. The <u>D°</u> <u>derivation</u> <u>basis</u>. For a positive function δ on R,

$\beta^{\circ}_{\delta} = \{(I,x)$: I is an interval in R, $x \in R$ and

$x \in I \subset (x-\delta(x), x+\delta(x))\}$ and

$D^{\circ} = \{\beta^{\circ}_{\delta}$: δ is a positive function on R}.

<u>Definition</u> 1.3. The <u>D</u> <u>derivation</u> <u>basis</u>. For a positive function δ on R,

$\beta_{\delta} = \{(I,x)$: I is an interval in R, $x \in R$, x is a left or right hand endpoint of I and $I \subset (x-\delta(x), x+\delta(x))\}$ and

$D = \{\beta_{\delta}$: δ is a positive function on R}.

<u>Definition</u> 1.4. A <u>partition</u> <u>π</u> <u>of</u> <u>an</u> <u>interval</u> [a,b] is a set

$\pi = \{(I_i,x_i)$: I_i is an interval in R, $x_i \in R$, $i=1,\ldots,n\}$

such that I_i, I_j are non-overlapping intervals for distinct i and j and the union of $\{I_i\}_{i=1}^{n}$ is [a,b].

<u>Definition</u> 1.5. Let π be a partition of [a,b]. Let g be a monotone increasing function where if $I_i = [a_i,b_i]$, then $g(I_i) = g(b_i) - g(a_i)$. Then for an arbitrary set $E \subset [a,b]$,

$m(\pi,E) = \Sigma_{i=1}^{n} \{g(I_i): (I_i,x_i) \in \pi, x_i \in E\}$.

<u>Definition</u> 1.6. For the $D^{\#}$ derivation basis, the D derivation basis, and the D° derivation basis, let

$\beta^{\#}_{\delta}\ [X] = \{(I,x) \in \beta^{\#}_{\delta} : x \in X\}$

$\beta^{\circ}_{\delta}\ [X] = \{(I,x) \in \beta^{\circ}_{\delta} : x \in X\}$

$\beta_{\delta}\ [X] = \{(I,x) \in \beta_{\delta} : x \in X\}$ and

$D^{\#}\ [X] = \{\beta^{\#}_{\delta}\ [X]: \beta^{\#}_{\delta} \in D^{\#}\}$

$D^{\circ}\ [X] = \{\beta^{\circ}_{\delta}\ [X]: \beta^{\circ}_{\delta} \in D^{\circ}\}$

$D\ [X] = \{\beta_{\delta}\ [X]: \beta_{\delta} \in D\}$.

Definition 1.7. For an arbitrary set $E \subset [a,b]$ and a monotone increasing function g,

$V(g,\ \beta^{\#}_{\delta}\ [E]) = \sup\ \{m(\pi,E),\ \pi \subset \beta^{\#}_{\delta}\}$

$V(g,D^{\#}[E]) = \inf\ \{V(g,\beta^{\#}_{\delta}[E]): \beta^{\#}_{\delta} \in D^{\#}\}$

and when the variation is viewed as a measure, we write this measure $g_{D^{\#}}[E]$ and $g_{D^{\#}}[E] = V(g,D^{\#}[E])$.

Definition 1.8. For an arbitrary set $E \subset [a,b]$ and a monotone increasing function g,

$V(g,\beta_{\delta}[E]) = \sup\ \{m(\pi,E),\ \pi \subset \beta_{\delta}\}$

$V(g,D[E]) = \inf\ \{V(g,\beta_{\delta}[E]): \beta_{\delta} \in D\}$

and write $g_{D}[E] = V(g,D[E])$.

Definition 1.9. For an arbitrary set $E \subset [a,b]$ and a monotone increasing function g,

$V(g,\beta^{\circ}_{\delta}[E]) = \sup\ \{m(\pi,E): \pi \subset \beta^{\circ}_{\delta}\}$

$V(g,D^{\circ}[E]) = \inf\ \{V(g,\beta^{\circ}_{\delta}[E]: \beta^{\circ}_{\delta} \in D^{\circ}\}$

and write $g_{D^{\circ}}[E] = V(g,D^{\circ}[E])$.

When $g(x) = x$, write $g_{D^{\circ}}[E]$ as $m_{D^{\circ}}[E]$.

Definition 1.10. For a non-negative point function f and a monotone increasing function g, let fg be the interval-point function $f(x)g(I)$. Let

$$V(fg,\beta_\delta^{\#}) = \sup \{\Sigma_\pi f(x_i)g(I_i) : \pi \subset \beta_\delta^{\#}\}$$

$$V(fg,D^{\#}) = \inf \{V(fg,\beta_\delta^{\#}) : \beta_\delta^{\#} \in D^{\#}\}$$

and write $g_D{}^{\#}(f) = V(fg,D^{\#})$.

It can be noted here that if f is non-negative, the $D^{\#}$ integral with respect to g can be viewed as the variation $g_D{}^{\#}(f)$. This will be shown later.

Definition 1.11. Local Character. A derivation basis B is said to have local character if for each $\beta_x \in B$, $x \in R$, there is a $\beta \in B$ such that $\beta[\{x\}] \subset \beta_x[\{x\}]$.

Definition 1.12. σ-local Character. A derivation basis B is said to have σ-local character if, for every sequence of disjoint sets $\{X_n\}$ in R and $\beta_n \in B$ there is a $\beta \in B$ such that $\beta[X_n] \subset \beta_n[X_n]$.

Definition 1.13. A derivation basis B is filtering down if when $\beta_1, \beta_2 \in B$, then there exists a $\beta_3 \in B$ such that $\beta_3 \subset \beta_1 \cap \beta_2$.

Definition 1.14. Ignores a point. A derivation basis B is said to ignore a point x if there is an element $\beta \in B$ for which there is no pair $(I,x) \in \beta$ for any I.

Definition 1.15. Let B be a derivation basis. Let $\mathcal{I}$ be a collection of intervals in R. A subset β^* of $\mathcal{I}$xR is said to be B-fine if for every $\beta \in B$ and every $x \in R$ either $\beta[\{x\}] = \Phi$ or else $\beta^* \cap \beta[\{x\}] \neq \Phi$.

The collection of all B-fine elements of $\mathcal{I}$xR is denoted as B^* and called the dual of B.

Definition 1.16. An interval-point function h(x,I) is B-integrable on the interval [a,b] if there exists a number c such that for any $\varepsilon > 0$ there exists a $\beta \in B$ such that $|\Sigma_\pi h(x,I) - c| < \varepsilon$ for any $\pi \subset \beta$ of the interval [a,b].

Note also the following observations that will be needed later.

Observation 1.1. Let D_o^* be the dual of D^o, then $\beta_o^* \in D_o^*$ if and only if β_o^* is a collection of pairs (I,x) and for each x and $\varepsilon > 0$ there is an $(I,x) \in \beta_o^*$ and $x \in I \subset (x-\varepsilon, x+\varepsilon)$.

Observation 1.2. The $D^\#$ derivation basis and the D^o derivation basis are filtering down and have σ- local character.

Proof. Filtering down clearly holds for both $D^\#$ and D^o. To see that both have σ- local character, let $\{X_n\}$ be a sequence of disjoint subsets of R and let $\{\beta^\#_{\delta_n}\} \subset D^\#$

or $\{\beta^{\circ}_{\delta_n} \subset D^{\circ}$.

Define $\delta(x) = \delta_n(x)$ for $x \in X_n$, $\delta(x) = 1$ otherwise.

Then $\beta^{\#}_{\delta}[X_n] \subset \beta^{\#}_{\delta_n}$ and similarly $\beta^{\circ}_{\delta}[X_n] \subset \beta^{\circ}_{\delta_n}$ for each n.

<u>Observation</u> 1.3. Note that the $D^{\#}$ derivation basis has local character. This is because for each $x \in R$ and $\beta^{\#}_{\delta_x} \in D^{\#}$, if $\delta(x) = \delta_x(x)$ then $\beta_{\delta} \subset \cup_{x \in R} \beta_{\delta_x}[\{x\}]$.

<u>Observation</u> 1.4. Since $D^{\#}$ derivation basis and D° derivation basis are filtering down and ignore no point, $D^{\#} \subset D^{*}_{\#}$ and $D^{\circ} \subset D^{*}_{\circ}$ [7, p.160]. Thus, $g_{D^{*}_{\#}}[E] \le g_{D^{\#}}[E]$ and $g_{D^{*}_{\circ}}[E] \le g_{D^{\circ}}[E]$.

<u>Observation</u> 1.5. If f is a non-negative point function which is $D^{\#}$ integrable with respect to g, then f is D° integrable with respect to g.

<u>Proof</u>. Let $\varepsilon > 0$ be given. Then, there exists $\delta: R \to R^{+}$ such that for all partitions $\pi \subset \beta^{\#}_{\delta}$

$|D^{\#}\text{-}\int f dg - \Sigma_{\pi} f(x_i) g(I_i)| < \varepsilon$.

In particular, for all partitions $\pi \subset \beta^{\#}_{\delta}$ with $x_i \in I_i$

$|D^{\#}\text{-}\int f dg - \Sigma_{\pi} f(x_i) g(I_i)| < \varepsilon$.

Therefore, f is D° integrable and $D^{\circ}\text{-}\int f dg = D^{\#}\text{-}\int f dg$.

The D integral, $D\text{-}\int f dx$ [7] is known to be the Perron integral as the $D^{\#}$ integral, $D^{\#}\text{-}\int f dx$ [7] is known to be the Lebesgue integral. Both derivation bases give rise to the

Lebesgue measure. However, not all derivation bases give rise to an outer measure. For example, the derivation basis that gives rise to the Riemann integral is not subadditive as an "outer content." [7]. When a derivation basis does give rise to a measure, the integral may not be the same as noted above.

Theorem 6.2 [7, p.158] of Thomson asserts that for a derivation basis B which is filtering down and has σ- local character then $V(g,B[X])$ gives rise to a true outer measure. This is an outer measure that is defined on all subsets of $[a,b]$ and must be (1) monotone on sets, (2) countably sub-additive, and (3) equal to 0 on the empty set (cf. Royden [5, p.53]). By Observation 1.2, the $D^{\#}$ derivation basis and the D° derivation basis give rise to true outer measures on $[a,b]$.

For each monotone function g, the $D^{\#}$ derivation basis and the D° derivation basis also generate an outer measure that satisfies the Carathèodory criterion [5, p.79] which states that if $A, B \subset [a,b]$ and $dist(A,B) > 0$ then the outer measure adds on A and B (i.e. $m^*(A\cup B) = m^*(A) + m^*(B)$). To see that the Carathéodory criterion is satisfied, let $\delta(x) = (1/3)dist(A,B)$ for all $x \in [a,b]$. Then the intervals in a partition π of $[a,b]$ which belong to $\beta_\delta^{\#}[A]$ do not meet those which belong to $\beta_\delta^{\#}[B]$. Therefore $V(g,\beta_\delta^{\#}[A\cup B]) = V(g,\beta_\delta^{\#}[A]) + V(g,\beta_\delta^{\#}[B])$ and $g_{D^{\#}}[A\cup B] = g_{D^{\#}}[A] + g_{D^{\#}}[B]$. By replacing $\beta_\delta^{\#}$ with β_δ°

in the above, it can be shown that D° satisfies the Caratheodory criterion. Therefore the $g_D\#$ and g_{D° measurable sets include the Borel sets.

The standard Lebesgue-Stieltjes outer measure associated with the monotone increasing function g is defined by $m_g^*[E] = \inf\{\Sigma_{i=1}^\infty [g(b_i) - g(a_i)]: E \subset \cup_{i=1}^\infty (a_i,b_i)\}$.

Now the equality of the $D^\#$ measure with the Lebesgue-Stieltjes measure will be established.

__Theorem__ 1.1. __Let__ $I = [a,b]$ __and let__ E __be any subset of__ I. __Suppose that__ g __is continuous from the left at a and continuous from the right at b, then__ $g_D\#[E] = m_g^*[E]$.

__Proof__. First we show that $g_D\#[E] \le m_g^*[E]$.

Let $\varepsilon > 0$ be given. Then, there exist an open set G with component intervals $\{(a_i,b_i)\}_{i=1}^\infty$ such that $E \subset G$ and $\Sigma_{i=1}^\infty [g(b_i)-g(a_i)] < m_g^*[E] + \varepsilon$.

Let $\delta(x)=1$ if $x \in G^c$ and let $\delta(x)=\mathrm{dist}(x,G^c)$ if $x \in G$.

Then, if $x \in E\cap(a_i,b_i)$ for i=1,2,...

$(x-\delta(x),x+\delta(x)) \subset (a_i,b_i)$.

Then, for any partition π of I, $\pi \subset \beta_\delta^\#$, if (I_j,x_j) is in $\pi[E]$ there is some i=1,2,... such that $I_j \subset (x_j-\delta(x_j),x_j+\delta(x_j)) \subset (a_i,b_i)$ which implies that $m(\pi,E) \le \Sigma_{i=1}^\infty [g(b_i)-g(a_i)]$.

Therefore $V(g,\beta_\delta^\#[E]) \le \Sigma_{i=1}^\infty [g(b_i)-g(a_i)]$.

This implies that $V(g,D^\#[E]) \le$

$$\Sigma_{i=1}^\infty [g(b_i)-g(a_i)] < m_g^*[E] + \varepsilon.$$

Hence $g_D\#[E] \le m_g^*[E]$.

Finally, we show that $m_g^*[E] \le g_D\#[E]$.

Let $\varepsilon > 0$ be given. Let $\delta:R\to R^+$ be such that

$V(g,\beta_\delta^\#[E]) < g_D\#[E] + \varepsilon$.

Then, $\cup_{x\in E}(x-\delta(x)/2, x+\delta(x)/2)$ is an open set.

Therefore, since g has at most countably many discontinuities, there exists half open intervals $\{I_n\}_{n=1}^\infty$, $I_n = [a_n,b_n)$, $I_n \cap I_m = \Phi$, $n\ne m$, where the endpoints of I_n are not discontinuities of g, and $\cup_{x\in E}(x-\delta(x)/2, x+\delta(x)/2) = \cup_{n=1}^\infty I_n$.

Since $\Sigma g(I_n) < \infty$, there exists $M > 0$ such that

$\Sigma_{n=M+1}^\infty g(I_n) < \varepsilon$.

Now $\cup_{n=1}^M I_n \subset \cup_{x\in E}(x-\delta(x), x+\delta(x))$.

Because g is continuous at b_n the intervals I_n' can be chosen such that $I_n' = [a_n,c_n]$, $c_n < b_n$ and

$\Sigma_{n=1}^M g(I_n) < \Sigma_{n=1}^M g(I_n') + \varepsilon$.

Therefore, $\cup_{n=1}^M I_n' \subset \cup_{x\in E}(x-\delta(x), x+\delta(x))$.

Since $\cup_{n=1}^M I_n'$ is compact, there exists $\{x_1,\ldots,x_p\} \subset E$ such that $\cup_{n=1}^M I_n' \subset \cup_{i=1}^p (x_i-\delta(x_i), x_i+\delta(x_i))$.

For the given $\delta(x)$, let π be a partition in $\beta_\delta^\#$ such that $\pi[E] = \{(y_r, I_r'')\}_{r=1}^s$

where $\{y_1,\ldots,y_s\} \subset \{x_1,\ldots,x_p\}$

and $\cup_{n=1}^M I_n' \subset \cup_{r=1}^s I_r''$ which is possible since

$I_n' \subset (x_i-\delta(x_i), x_i+\delta(x_i))$ for some $i=1,\ldots,p$.

Then, $\Sigma_{n=1}^\infty g(I_n) < \Sigma_{n=1}^M g(I_n') + 2\varepsilon \le \Sigma_{r=1}^s g(I_r'') + 2\varepsilon$

which implies that $m_g^*[E] \le \Sigma_{n=1}^\infty g(I_n) + \varepsilon < \Sigma_{r=1}^s g(I_r'') + 3\varepsilon <$

$$V(g,\beta_\delta^\#[E]) + 3\varepsilon < g_D\#[E] + 4\varepsilon.$$

Hence, $m_g^*[E] \le g_D\#[E]$.

Thus, $m_g^*[E] = g_D\#[E]$.

We now show that for continuous, monotone increasing functions g, the Lebesgue-Stieltjes measure is also equal to the measure generated by the dual of D°.

Theorem 1.2. For a monotone increasing function g and for each set E, $g_{D_\circ^*}[E] = m_g^*[E]$ if and only if g is continuous.

Proof. Assume g is continuous on $[a,b]$. By a theorem in Saks [6, p.100], when g is continuous $m_g[E] = |g[E]|$. Thomson shows [7, p.162] that $|g[E]| = m_{D_\circ^*}[g[E]]$.

where $|g[E]|$ is the Lebesgue measure of the set $g[E]$ and $m_{D_\circ^*}[g[E]]$ is the measure of $g[E]$ with respect to the dual of the $D_\circ$ derivation basis.

So it remains to show that $m_{D_\circ^*}[g[E]] = g_{D_\circ^*}[E]$.

For $\beta^* \in D_\circ^*$, it follows from Observation 1.1 and the continuity of g that

$\{(x,[u,v]): (g(x),[g(u),g(v)] \in \beta^*\}$ is an element of $D_\circ^*$

and that

$\{(g(x),[g(u),g(v)]): (x,[u,v]) \in \beta^*\}$ is an element of $D_\circ^*$.

We will denote them by $g^{-1}(\beta^*)$ and $g(\beta^*)$ respectively.

Let $\varepsilon > 0$ be given and $\beta^* \in D_\circ^*$ such that

(*) $V(1,\beta^*[g[E]] < m_{D_\circ^*}[g[E]] + \varepsilon.$

Therefore by the definition of $V(1,\beta^*[g[E]])$,

$\{\Sigma_{i=1}^n [g(b_i)-g(a_i)]: (g(x_i),[g(a_i),g(b_i)]) \in \beta^*, g(x_i) \in g[E]\}$
$\leq V(1,\beta^*[g[E]]).$

Hence,

$$(**)\ \sup\{\Sigma[g(b_i)-g(a_i)]:\ (x_i,[a_i,b_i]) \in g^{-1}(\beta^*),\ x_i \in E\} = V(g,g^{-1}(\beta^*)[E])$$
$$\le V(1,\beta^*[g[E]]).$$

Since the supremum is greater than or equal to $g_{D_o}^*[E]$,
$g_{D_o}^*[E] \le m_{D_o}^*[g[E]] + \varepsilon$ by (*) and (**)
and hence $g_{D_o}^*[E] \le m_{D_o}^*[g[E]]$.
We now prove the reverse inequality.
Let $\beta^* \in D_o^*$ such that $V(g,\beta^*[E]) < g_{D_o}^*[E] + \varepsilon$.

By definition of β^*, for each x and $\varepsilon > 0$, there exists an I such that $|I| < \varepsilon$ with $x \in I$ and $(I,x) \in \beta^*$. Therefore

there is going to be a $g[\beta_1^*] \in D_o^*$ such that for each $\delta > 0$, there exists $(I,x) \in \beta^*$ such that $g[I] < \delta$
by the continuity of g.
Since $g[\beta_1^*] = \beta'$ may not include all $(g(x),g[I])$ such that $(I,x) \in \beta^*$, there will be fewer finite sums, so
$V(1,\beta'[g[E]]) \le V(g,\beta^*[E]) < g_{D_o}^*[E] + \varepsilon$.
Hence $m_{D_o}^*[g[E]] \le g_{D_o}^*[E] + \varepsilon$.
Therefore $m_{D_o}^*[g[E]] \le g_{D_o}^*[E]$.
Thus $m_{D_o}^*[g[E]] = g_{D_o}^*[E]$.
Therefore, when g is continuous, $g_{D_o}^*[E] = m_g[E]$.
Assume g is monotone on $[a,b]$ and not continuous.
Without loss of generality, assume g is discontinuous from

the left at x_0. Let $\varepsilon < [g(x_0)-g(x_0-)]$.

Let E be a set with $x_0 \in E$. Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of open intervals such that

$E \subset \bigcup_{n=1}^{\infty} I_n$ and $\Sigma_{n=1}^{\infty} g(I_n) < m_g[E] + \varepsilon$.

Let β^* be the dual such that if $x \in I_n$ and $x \neq x_0$,

$(I,x) \in \beta^*$ if and only if $I \subset I_n$ and $x_0 \notin I$,

and for x_0, let $(I,x_0) \in \beta^*$ if and only if

I is contained in the I_n to which x_0 belongs and x_0 is a left hand endpoint of I.

Let $\{(x_i,J_i)\}_{i=1}^{n} \subset \beta^*[E]$.

Then $\Sigma_{i=1}^{n} g(J_i) + [g(x_0)-g(x_0-)] < \Sigma_{n=1}^{\infty} g(I_n)$.

Therefore $V(g,\beta^*[E]) + [g(x_0)-g(x_0-)] \leq \Sigma_{n=1}^{\infty} g(I_n)$.

Hence $g_{D_0^*}[E] + [g(x_0)-g(x_0-)] \leq m_g[E] + \varepsilon$.

So $g_{D_0^*}[E] < m_g[E]$ by the choice of ε.

Since the dual derivation basis D_0^* does not have the partitioning property, D_0^* does not give rise to an integral. However, the $D^{\#}$ derivation basis and the D° derivation basis do.

By Observation 1.5, if f is $D^{\#}$ integrable, it is D° integrable. It is known [1] that if f is D° integrable, then f is Perron-Stieltjes integrable. Therefore, if f is $D^{\#}$ integrable, it is measurable with respect to the Lebesgue-Stieltjes measure.

Also, by Corollary 6.6 of McShane [2, p.16], if f is $D^{\#}$ integrable, it is absolutely integrable.

Therefore, it suffices to show equivalence of the $D^{\#}$ Stieltjes integral with the Lebesgue-Stieltjes integral for non-negative functions f.

The following lemmas are needed to establish that whenever a non-negative function f is Lebesgue-Stieltjes integrable with respect to a monotone g, it is $D^{\#}$ integrable with respect to g and the integrals agree.

__Lemma__ 1.1. __Let__ E __be__ __any__ __Lebesgue-Stieltjes__ __measurable__ __set__. __Then__ __the__ __characteristic__ __function__ __of__ E, $\chi_E(x)$, __is__ $D^{\#}$ __integrable__ __with__ __respect__ __to__ g __and__

$$D^{\#}\text{-}\int\chi_E(x)dg(x) = m_g[E].$$

__Proof__. Let the interval-point function h(x,I) be defined as $\chi_E(x)g(I)$. Then $\Sigma_\pi\chi_E(x_i)g(I_i) = m[\pi,E]$.

Let $\varepsilon > 0$ be given. Then, there exists a closed set F such that $F \subset E$ and $m_g[E] < m_g[F] + \varepsilon$.

Let $\delta:R\to R^+$ be such that

$$(x-\delta(x),x+\delta(x)) \subset [a,b]\setminus F \text{ if } x \in [a,b]\setminus F$$

and $V(g,\beta^{\#}_\delta[E]) < g_{D^{\#}}[E] + \varepsilon$. Let $\pi \subset \beta^{\#}_\delta$ be a partition of [a,b]. Then, $\pi[F] = \{(x_i,I_i)\}_{i=1}^n$, where $F \subset \bigcup_{i=1}^n I_i$.

So, $m_g[E] < m_g[F]+\varepsilon \le \Sigma_{\pi[F]}\chi_F(x)g(I)+\varepsilon \le \Sigma_{\pi[E]}\chi_E(x)g(I)+\varepsilon =$

$$m[\pi,E]+\varepsilon \le V(g,\beta^{\#}_\delta[E])+\varepsilon < g_{D^{\#}}[E] + 2\varepsilon = m_g[E] + 2\varepsilon.$$

Therefore $|\Sigma_\pi\chi_E(x)g(I)-m_g[E]| < \varepsilon$ for all $\pi \subset \beta^{\#}_\delta$.

Hence χ_E is $D^{\#}$ integrable and $D^{\#}\text{-}\int\chi_E(x)dg(x) = m_g[E]$.

__Lemma__ 1.2. __Let__ s(x) __be__ __any__ __Lebesgue-Stieltjes__ __simple__ *function with respect to g. That is, s(x) is measurable with respect to m_g and takes on a finite number of values. Then s(x) is $D^{\#}$ integrable.*

Proof. Since $s(x)= \Sigma_{n=1}^{m} c_n \chi_{E_n}(x)$ where χ_{E_n} is the characteristic function of a Lebesgue-Stieltjes measurable set and the $D^{\#}$ Stieltjes integral is finitely additive on functions [2], s(x) is $D^{\#}$ integrable.

Theorem 1.3. If $f \geq 0$ is Lebesgue-Stieltjes integrable with respect to g, then f is $D^{\#}$ integrable with respect to g.

Proof. Since f is Lebesgue-Stieltjes integrable with respect to g, $L\text{-}S\int f dg < \infty$.

Also, there exists a sequence of simple functions $\{s_n\}$ such that $s_n(x) \leq s_{n+1}(x)$ for all n

and $s_n(x) \to f(x)$ as $n \to \infty$.

By Theorem 7.6 of McShane [2, p.20] and the well-known monotone convergence theorem for Lebesgue-Stieltjes integrals [5], f is $D^{\#}$ integrable

and $D^{\#}\text{-}\int f dg = L\text{-}S\int f dg$.

By Lemma 1.2, Theorem 7.6 of McShane [2, p.20], and the fact that if f is $D^{\#}$ integrable with respect to g, it is measurable with respect to the Lebesgue-Stieltjes measure m_g, f is therefore the limit of simple functions and hence Lebesgue-Stieltjes integrable.

Theorem 1.4. *If* $f \geq 0$ *and* $D^{\#}$ *integrable, and* $[a,b]$ *is any interval where* $g(a) = g(a-)$ *and* $g(b) = g(b+)$, *then* $D^{\#}\text{-}\int f dg = g_{D^{\#}}(f)$ *on* $[a,b]$.

Proof. Let $\varepsilon > 0$ be given. Let $[a,b]$ be any interval where $g(a) = g(a-)$ and $g(b) = g(b+)$.

Then, $g_{D^{\#}}[a,b] = m_g[a,b] = g(b)-g(a)$.

Then, there exists a δ_1 such that

$$|D^{\#}\text{-}\int f dg - \Sigma_{\pi} f(x_i)g(I_i)| < \varepsilon$$

for all partitions $\pi \subset \beta^{\#}_{\delta_1}$ of $[a,b]$.

There exists a δ_2 such that $V(fg,\beta^{\#}_{\delta_2}) < V(fg,D^{\#}) + \varepsilon$.

Let $\delta(x) = \min\{\delta_1(x),\delta_2(x)\}$.

Then, $V(fg,\beta^{\#}_{\delta}) < V(fg,D^{\#}) + \varepsilon$.

Therefore, there exists a partition π of $[a,b]$ in $\beta^{\#}_{\delta}$ such that $-\varepsilon+V(fg,D^{\#}) \leq -\varepsilon+V(fg,\beta^{\#}_{\delta}) < \Sigma_{\pi} f(x_i)g(I_i) <$

$$V(fg,\beta^{\#}_{\delta}) < V(fg,D^{\#}) + \varepsilon.$$

Hence $|D^{\#}\text{-}\int f dg - V(fg,D^{\#})| \leq$

$$|D^{\#}\text{-}\int f dg-\Sigma_{\pi} f(x_i)g(I_i)|+|\Sigma_{\pi} f(x_i)g(I_i)-V(fg,D^{\#})| < 2\varepsilon.$$

Thus, for any non-negative point function f on an interval $[a,b]$ where $g(a) = g(a-)$ and $g(b) = g(b+)$,

$$D^{\#}\text{-}\int f dg = g_{D^{\#}}(f).$$

Because a function of bounded variation can be expressed as the difference of two monotone increasing functions, in order to have equality of the integrals for functions g of bounded variation it suffices to show equality of the integrals for monotone functions g.

This is what was just done.

The Lebesgue-Stieltjes measure of an interval $[a,b]$ with respect to monotone increasing g is $g(b+)-g(a-)$ and thus the measure and the integral are affected by points outside of $[a,b]$. On the other hand, the $D^{\#}$ Stieltjes integral on $[a,b]$ is not affected by points outside of $[a,b]$. However,

$$L\text{-}S_a\int^b fdg - f(a)[g(a)-g(a-)] - f(b)[g(b+)-g(b)]$$

where the right hand side is a modification of the Lebesgue-Stieltjes integral indicated by Saks [6, p.208]. This integral is additive on intervals and is equal to the $D^{\#}$ integral on $[a,b]$.

Similar proofs to the above can be used to show that the weak Kurzweil base of Definition 2.2.1 and the weak Kempisty q-base of Definition 2.2.4 in K.M. Ostaszewski's Ph.D. Dissertation [4] give rise to Stieltjes integrals that are the Lebesgue-Stieltjes integral in the plane [3]. These definitions and proofs can be generalized to R^n.

This material is part of my dissertation [3] which was prepared under the direction of Professor J. Foran.

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