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ON A RESULT OF S. KUREPA

Introduction

In an article published in 1956, S. Kurepa [2] proved the following theorem.

Theorem. *There exist Lebesgue measurable sets* $A, B \subset R^n$ *such that the set* $A + B = \{a + b: a \in A, b \in B\}$ *is nonmeasurable.*

Here a + b is the ordinary coordinate wise sum of a and b, i.e. if $a = (a_1, a_2, \ldots, a_n)$ and $b = b_1, b_2, \ldots, b_n)$ then $a+b = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$.

The proof of this theorem can be found in M. Kuczma's new book "An Introduction to the Theory of Functional Equations and Inequalities" ([1], pg. 256). Kuczma introduces Kurepa's theorem, saying it "shows a certain irregularity of the operation +". The purpose of this paper is to extend Kurepa's result by showing that a wide class of operations on R^n (i.e. functions on $R^n \times R^n$ into R^n) actually share the irregularity of the operation + noted above.

Before presenting our results we mention that the sets A and B in Kurepa's paper (and in Kuczma's book) are constructed using a measurable Hamel basis and that this construction can not be extended to show a similar result for operations different from +. Furthermore, Kurepa's sets A and B both turn out to be sets of Lebesgue measure zero.

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In this paper, $N(a,r)$ will denote the open ball in R^n with center a and radius r. Furthermore, a set $A \subset R^n$ is called a *universal null set*, if $\mu(A) = 0$ for each complete measure space (R^n, M_μ, μ) that satisfies: $N(a,r) \in M_\mu$ for each $a \in R^n$ and each $r > 0$ and $\lim_{r \to 0^+} \mu(N(a,r)) = 0$ for each $a \in R^n$. A point $c \in R^n$ will be called rational if all its coordinates are rational numbers.

Results. The following lemma will be used in the proofs of both of our theorems.

Lemma 1. Suppose f is a function on $R^n \times R^n$ into R^n and N, X and Y are open balls in R^n. If $C = \{x_\tau | \tau < \omega_c\} \subset X$ and $D = \{y_\tau | \tau < \omega_c\} \subset Y$ and $N = \{t_\tau | \tau < \omega_c\}$, where ω_c denotes the least ordinal having the cardinality of the continuum, satisfy the following conditions:

(i) $f(x_\tau, y_\tau) = t_\tau$ for $\tau < \omega_c$.

(ii) $x_\sigma \neq x_\tau$, $y_\sigma \neq y_\tau$ and $t_\sigma \neq t_\tau$ if $1 \leq \sigma < \tau < \omega_c$.

(iii) For each $t \in N$ there is a unique one to one function $h_t\colon X \to Y$ such that $f(x, h_t(x)) = t$ for all $x \in X$.

Then there exist sets A and B such that $A \subset C$ and $B \subset D$ and $f(A \times B)$ is nonmeasurable in the sense of Lebesgue.

Proof. The collection of all uncountable closed subsets of N has cardinality of the continuum, this collection can be written in the form $\{F_\alpha\colon \alpha < \omega_c\}$. We will make repeated use of the fact that each F_α, $\alpha < \omega_c$, has cardinality of the continuum.

Pick f_{11}, f_{12}, distinct elements from F_1. By the properties of the sets C and D there exists a σ_1, $\sigma_1 < \omega_c$, such that $f(x_{\sigma_1}, y_{\sigma_1}) = f_{11}$. Set $a_1 = x_{\sigma_1}$ and $b_1 = y_{\sigma_1}$.

By the hypothesis on f, the set $\{\sigma\colon \sigma < \omega_c$ and either

$f(a_1, y_\sigma) = f_{12}$ or $f(x_\sigma, b_1) = f_{12}\}$ contains at most two elements. Therefore, again by the properties of the sets C and D, there exists a σ_2, $\sigma_2 < \omega_c$ such that $f(x_{\sigma_2}, y_{\sigma_2}) \in F_2$ and $f_{12} \notin \{f(x_{\sigma_i}, y_{\sigma_j}) : i, j, \in \{1,2\}\}$.

Set $a_2 = x_{\sigma_2}$ and $b_2 = y_{\sigma_2}$ and denote $f(a_2, b_2)$ by f_{21}. Clearly, there exists an element, say f_{22}, in F_2 such that $f_{22} \notin \{f(a_i, b_j) : i, j \in \{1,2\}\}$.

We proceed by transfinite induction. Suppose α is an ordinal number, $\alpha < \omega_c$, and that for each $\beta < \alpha$, we have selected points $a_\beta, b_\beta, f_{\beta 1}, f_{\beta 2}$ in R^n satisfying:

(o) $a_\beta \in C$ and $b_\beta \in D$ for each $\beta, \beta < \alpha$,

(p) $f(a_\beta, b_\beta) = f_{\beta 1}$ and $f_{\beta 1} \in F_\beta$, for each $\beta, \beta < \alpha$,

(q) $f(a_\gamma, b_\delta) \neq f_{\beta 2}$ for each γ, δ, β; $\gamma, \delta, \beta < \alpha$ and $f_{\beta 2} \in F_\beta$, for each $\beta, \beta < \alpha$.

By the hypotheses on f and the fact that the cardinal of α is less than that of the continuum it follows that the set

$$\bigcup_{\sigma, \gamma < \alpha} \{\delta : \delta < \omega_c \text{ and either } f(a_\sigma, y_\delta) = f_{\gamma 2} \text{ or } f(x_\delta, b_\sigma) = f_{\gamma 2}\}$$

has cardinality less than that of the continuum.

Therefore, by the properties of C and D, there exists a $\sigma_\alpha, \sigma_\alpha < \omega_c$ such that $f(x_{\sigma_\alpha}, y_{\sigma_\alpha}) \in F_\alpha$ and $f(x_{\sigma_\alpha}, b_\beta) \neq f_{\gamma 2}$ and $f(a_\beta, y_{\sigma_\alpha}) \neq f_{\gamma 2}$ for every β, γ; $\beta, \gamma < \alpha$. Set $a_\alpha = x_{\sigma_\alpha}$ and $b_\alpha = y_{\sigma_\alpha}$ and denote $f(a_\alpha, b_\alpha)$ by $f_{\alpha 1}$; i.e. $f_{\alpha 1} = f(a_\alpha, b_\alpha)$.

Clearly, again as the cardinality of α is less than that of the continuum, there exists an element, say $f_{\alpha 2}$, in F_α such that $f_{\alpha 2} \notin \{f(a_\gamma, b_\delta) : \gamma, \delta \leq \alpha\}$.

Therefore, by transfinite induction, we conclude that there exist four transfinite sequences $\{a_\alpha\}_{\alpha < \omega_c}$, $\{b_\alpha\}_{\alpha < \omega_c}$, $\{f_{\alpha 1}\}_{\alpha < \omega_c}$ and $\{f_{\alpha 2}\}_{\alpha < \omega_c}$ satisfying:

(O) $\quad a_\alpha \in C$ and $b_\alpha \in D$ for each α, $\alpha < \omega_c$,

(P) $\quad f(a_\alpha, b_\alpha) = f_{\alpha 1} \in F_\alpha$ for each α, $\alpha < \omega_c$,

(Q) $\quad f(a_\alpha, b_\beta) \neq f_{\gamma 2}$ for each α, β, γ; $\alpha, \beta, \gamma < \omega_c$ and $f_{\gamma 2} \in F_\gamma$, for each γ, $\gamma < \omega_c$.

Set $A = \{a_\alpha : \alpha < \omega_c\}$ and $B = \{b_\alpha : \alpha < \omega_c\}$, Then, clearly, $f(A \times B)$ is nonmeasurable; in fact $f(A \times B)$ has outer Lebesgue measure equal to $m(N)$ and inner Lebesgue measure equal to zero. This completes the proof of Lemma 1.

Martin's Axiom, which is weaker than the continuum hypothesis, implies that the union of less than c, the cardinal of the continuum, first category sets is a first category set and that the union of less than c sets of measure zero is a set of measure zero (see the following: D.A. Martin and R.M. Solovay, Internal Cohen extensions, Ann. Math. Logic 2 (1970), 143-178). Moreover, the hypothesis "the union of less than c first category sets is first category and the union of less than c sets of measure zero is a set of measure zero" is even weaker than Martin's axiom. For the purpose of reference, let (F) denote the hypothesis "the union of less than c first category sets is first category and the union f less than c sets of measure zero is a set of measure zero".

Our first theorem shows, assuming (F), that for each function f in a certain wide class of functions on $R^n \times R^n$ into R^n, including "+", there exists a pair of universal null sets A and B such that $f(A \times B) = \{f(a,b) : (a,b) \in A \times B\}$ is nonmeasurable in the sense of Lebesgue.

Theorem 1. Let $f: R^n \times R^n \to R^n$. *Suppose* X, Y *and* N *are open balls in* R^n *satisfying the following conditions*:

(i) For each $t \in N$ there is a unique function h_t: $X \to Y$ such that $f(x, h_t(x)) = t$ for all $x \in X$.

(ii) For each $t \in N$, h_t is a homeomorphism of X into Y.

Then, assuming (F), there exists a pair of universal null sets A and B such that $f(A \times B)$ is Lebesgue nonmeasurable.

Proof. Let D_1 and D_2 denote respectively the rational points in X and Y. Let $\mathcal{R}$ and $\mathcal{S}$ denote respectively, the collections of all open sets containing D_1 and D_2, which are subsets of X and Y respectively. It is an easy exercise to show that $\mathcal{R}$ and $\mathcal{S}$ have cardinality of the continuum. Let ω_c denote the least ordinal number having cardinality of the continuum. Then the collections $\mathcal{R}$ and $\mathcal{S}$ can be written as transfinite sequences: $\{R_\tau: \tau < \omega_c\}$ and $\{S_\tau: \tau < \omega_c\}$. Also, the set N can be written in the form $N = \{t_\tau: \tau < \omega_c\}$.

We will now choose, using transfinite induction, two transfinite sequences

$$\{x_\tau: \tau < \omega_c\} \text{ and } \{y_\tau: \tau < \omega_c\}.$$

We take x_1, y_1 to be any two points such that $x_1 \in R_1$, $y_1 \in S_1$ and $f(x_1, y_1) = t_1$. Such a pair exists, as $X \setminus R_1$ and $Y \setminus S_1$ are both nowhere dense and h_{t_1} is a homeomorphism of X onto $h_{t_1}(X)$, which is a subset of Y.

Now, suppose that τ is any ordinal number less than ω_c and that x_σ and y_σ have been chosen for all ordinals σ less than τ, in such a way that:

(1) $$x_\sigma \in \bigcap_{\alpha \leq \sigma} R_\alpha \text{ and } y_\sigma \in \bigcap_{\alpha \leq \sigma} S_\alpha \text{ for all } \sigma, \alpha < \tau.$$

(2) $$f(x_\sigma, y_\sigma) = t_\sigma \text{ for all } \sigma, \sigma < \tau.$$

(3) $$x_\alpha \neq x_\sigma \text{ and } y_\alpha \neq y_\sigma \text{ if } 1 \leq \alpha < \sigma < \tau.$$

We now proceed to select an appropriate pair x_τ, y_τ. To do this consider the sets

$$E_\tau = \bigcup_{\sigma \leq \tau} (X \setminus R_\sigma) \cup \bigcup_{\sigma < \tau} \{x_\sigma\} \text{ and}$$

$$F_\tau = \bigcup_{\sigma \leq \tau} (Y \setminus S_\sigma) \cup \bigcup_{\sigma < \tau} \{y_\sigma\}.$$

E_τ and F_τ are unions of less than c nowhere dense sets. Therefore, assuming (F), both are sets of the first category.

Therefore, as argued in the $\tau = 1$ case, since h_{t_τ} is a homeomorphism of X onto $h_{t_\tau}(X)$, there exists

$$x_\tau \in X \setminus E_\tau, \text{ such that } h_{t_\tau}(x_\tau) = y_\tau \in Y \setminus F_\tau.$$

Therefore, by transfinite induction, we obtain two complete transfinite sequences

$$\{x_\tau : \tau < \omega_c\} \text{ and } \{y_\tau : \tau < \omega_c\} \text{ satisfying:}$$

(I) $$x_\tau \in \bigcap_{\sigma \leq \tau} R_\tau \text{ and } y_\tau \in \bigcap_{\sigma \leq \tau} S_\tau \text{ for all } \tau, \tau < \omega_c.$$

(II) $$f(x_\tau, y_\tau) = t_\tau \text{ for all } \tau, \tau < \omega_c.$$

(III) $$x_\alpha \neq x_\sigma \text{ and } y_\alpha \neq y_\sigma \text{ if } 1 \leq \alpha < \sigma < \omega_c.$$

Set $C = \{x_\tau : \tau < \omega_c\}$ and $D = \{y_\tau : \tau < \omega_c\}$. Then $f(C \times D) \supset \{f(x_\tau, y_\tau) : \tau < \omega_c\} = \{t_\tau : \tau < \omega_c\} = N$.

Suppose (R^n, M_μ, μ) is any complete measure space that satisfies: $N(\bar{a}, \bar{r}) \in M_\mu$ for each $\bar{a} \in R^n$ and each $\bar{r} > 0$ and $\lim_{\bar{r} \to 0+} \mu(N(\bar{a}, \bar{r})) = 0$ for each $\bar{a} \in R^n$. The set D_1 can be written in the form $D_1 = \{u_n : n = 1,2,\ldots\}$. Let $\epsilon > 0$ be given. For each n, there exists an open ball B_n, $B_n \subset X$, such that $u_n \in B_n$ and $\mu(B_n) < \epsilon/2^n$. Let $G = \bigcup_{n=1}^{\infty} B_n$. Then $G \in \mathcal{R}$ and $\mu(G) < \epsilon$. Therefore $G = R_\tau$ for some $\tau, \tau < \omega_c$. By (I), this implies that $x_\alpha \in G$ for each $\alpha, \tau \leq \alpha < \omega_c$, which in turn implies that

$$C \setminus G \subset \{x_\alpha : \alpha < \tau\},$$

which, by (F), is a set of measure zero.

Therefore $\mu(C) \leq \mu(G) + +\mu(C \setminus G) < \epsilon + 0$. Hence C is a universal null set. A similar argument shows that D is a universal null set.

By the definitions of C and D and the hypotheses on f it immediately follows, using Lemma 1, that there exist sets A and B such that $A \subset C$, $B \subset D$ and $f(A{\times}B)$ is nonmeasurable. A and B are universal null sets since they are subsets of C and D respectively.

Remark 1. Clearly the operation +, i.e. the function f defined by the formula $f(x,y) = x+y$ for every $x, y \in R^n$, satisfies the conditions of Theorem 1 and therefore, assuming (F), there exists a pair of universal null sets A and B in R^n such that A+B is nonmeasurable. Clearly coordinate-wise multiplication also works.

The purpose of our second theorem is to extend Kurepa's theorem without using (F). The following lemma, which extends a result of Utz [6], will be used in the proof of Theorem 2.

Lemma 2. Suppose s is a real number and $s \neq 0$. Suppose further that $\{f_t \mid t \in M\}$ is a collection of functions on R into R satisfying:

(*) There exists a v, $v > 0$ and $\bar{x}$, $\bar{y} \in R$ such that $f_t(\bar{x}) \in N(\bar{y}, v/2)$

for every $t \in M$ and $f_t'(x) \in (s-|s|/10, s+|s|/10)$ for every $t \in M$ and every $x \in [\bar{x}-v, \bar{x}+v]$.

Then there exists a compact set C, $C \subset [\bar{x}-v, \bar{x}+v]$, of measure zero and a compact set D, $D \subset [\bar{y}-v, \bar{y}+v]$, of measure zero such that the cardinality of (CxD) $\cap$ (the graph of f_t) is c for each $t \in M$.

Proof. If n_o, $n_o > 2$, is a sufficiently large natural number, then the cardinality of (CxD) $\cap$ (the graph of f_t) is c for each $t \in M$, where C is the "Cantor-like" subset of $[\bar{x}-v, \bar{x}+v]$ formed by taking out the middle n_o^{th} open intervals at each step of the "Cantor-like" construction and D is the "Cantor-like" subset of $[\bar{y}-v, \bar{y}+v]$ formed by taking out the middle n_o^{th} open intervals at each stage. We remark that C and D are compact sets of Lebesgue measure zero. This fact is proved, using the "nested square theorem", by showing that if n_o is sufficiently large and $t \in M$ is given, then there exists a stage n_1 in the construction of CxD such that the graph of f_t intersects the interiors of at least two, call them C_{11}, C_{12}, of the 4^{n_1} squares in this stage of the construction. There exists a stage n_2, $n_2 > n_1$ such that f_t intersects the interiors of at least two, call them C_{111}, C_{112}, of the subsquares of C_{11} in this stage of the construction of CxD. Similarly, there is a stage $n_2' > n_1$ such that f_t intersects the interior of at least two, call them C_{121}, C_{122}, of the subsquares of C_{12} in this stage of the construction. Continuing in this way, for each sequence $\{m_i\}_{i=1}^{\infty}$, where $m_i \in \{1,2\}$, we get a nested sequence of squares, namely C_{m_1}, $C_{m_1m_2}$, $C_{m_1m_2m_3}, \ldots$, whose intersection yields a point in the set (CxD) $\cap$ (the graph of f_t). Since there are c such that sequences $\{m_i\}_{i=1}^{\infty}$, and each yields a different point, we obtain our result, i.e. the cardinality of $(C\times D) \cap$ (the graph of f_t) is c for each $t \in M$.

Theorem 2. *Suppose* $f = (f_1, f_2, \ldots, f_n): R^n \times R^n \to R^n$ *satisfies the following conditions.*

(α) *The* $2n^2$ *partial derivatives* (n *functions and* 2n *variables*) *exist and are continuous in some neighbourhood of* $(x_0, y_0) \in R^n \times R^n$.

(β) $\begin{vmatrix} D_1 f_1 & \cdots & D_n f_1 \\ \vdots & & \vdots \\ D_1 f_n & \cdots & D_n f_n \end{vmatrix} (x_0, y_0) \neq 0$ *and* $\begin{vmatrix} D_{n+1} f_1 & \cdots & D_{2n} f_1 \\ \vdots & & \vdots \\ D_{n+1} f_n & \cdots & D_{2n} f_n \end{vmatrix} (x_0, y_0) \neq 0.$

Then there exist measurable sets $A, B \subset R^n$ *such that* $m(A) = m(B) = 0$ *and* $f(A\times B)$ *is nonmeasurable.*

Proof. f can be viewed as an nx1 column matrix. The nxn matrices $\frac{\partial f}{\partial x} = \left[\frac{\partial f}{\partial x_1}\ \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\right]$ and $\frac{\partial f}{\partial y} = \left[\frac{\partial f}{\partial y_1}\ \frac{\partial f}{\partial y_2} \cdots \frac{\partial f}{\partial y_n}\right]$ are both invertible at (x_o, y_o) by the hypotheses of this theorem. By the implicit function theorem, there is a continuously differentiable function $g(x,t)$, defined for x near x_o and t near $f(x_o, y_o)$ such that $f(x, g(x,t)) = t$. By implicit differentiation, we have $\frac{\partial g}{\partial x} = -\left[\frac{\partial f}{\partial y}\right]^{-1} \left[\frac{\partial f}{\partial x}\right]$. Therefore $\frac{\partial g}{\partial x}$ is invertible at the point (x_o, t_o), where $t_o = f(x_o, y_o)$. Since $\frac{\partial g}{\partial x} = \left[\frac{\partial g}{\partial x_1}\ \frac{\partial g}{\partial x_2} \cdots \frac{\partial g}{\partial x_n}\right]$ and $g = (g_1, g_2, \ldots, g_n)$ it follows that there exist $i, j \in \{1, 2, \ldots, n\}$ such that $\frac{\partial g_j}{\partial x_i}(x_o, t_o) = s$ and $s \neq 0$.

By the implicit function theorem there exist X, Y and N, open balls in R^n, containing x_o, y_o and t_o respectively, such that:

For each $t \in N$ and $x \in X$, $g(x,t)$ is the unique element in Y satisfying $f(x, g(x,t)) = t$.

Since g is continuously differentiable, there exists a positive real number v and a neighborhood M of t_o, $M \subset N$ such that:

$g_j(x_o,t) \in N(\hat{y},v/2)$ for every $t \in M$ and $\frac{\partial g_j(x,t)}{\partial x_i} \in (s-|s|/10,$ $s+|s|/10)$ for each $t \in M$ and each $x \in \overline{N}_i(x_o,v)$, where $\hat{y}(\in R)$ denotes the j^{th} component of y_o, $s = \frac{\partial g_j}{\partial x_i}(x_o,t_o)$ and $\overline{N}_i(x_o,v) = \{x \in \overline{N}(x_o,v) |$ the k^{th} component of x is equal to the k^{th} component of x_o for each k different from i$\}$, where $\overline{N}(x_o,v)$ is the closed ball with center x_o and radius v.

Therefore, by Lemma 2 there exist sets C and D such that: $C \subset \overline{N}_i(x_o,v)$, $D \subset \overline{N}(\hat{y},v)$ and C and D are compact sets, both having one dimensional Lebesgue measure zero, and

(the graph of g_{tj}) $\cap$ (CxD) has cardinality c for each $t \in M$, where $g_{tj}(x) = g_j(x,t)$ for each x.

Furthermore, $g_{tj}\Big|_{\overline{N}_i(x_o,v)}$ is a 1 to 1 function for each $t \in M$.

Therefore, if $t \in M$, there exist two transfinite sequences $\{c^t_\tau\}_{\tau < \omega_c}$ and $\{d^t_\tau\}_{\tau < \omega_c}$ such that: $c^t_\tau \in C$ and $d^t_\tau \in R \times R \times \ldots \times R \times D \times R \times \ldots R$ (where we have n factors and D appears in the j^{th} place) for each $\tau < \omega_c$; $c^t_\tau \neq c^t_{\tau'}$, $d^t_\tau \neq d^t_{\tau'}$, for each $\tau, \tau' < \omega_c$, $\tau \neq \tau'$ and $f(c^t_\tau, d^t_\tau) = t$ for each $\tau < \omega_c$.

It now follows, by a simple argument involving transfinite induction, that there exist two transfinite sequences $\{e_\tau\}_{\tau < \omega_c}$ and $\{f_\tau\}_{\tau \leq \omega_c}$ such that: $e_\tau \neq e_{\tau'}$ and $f_\tau \neq f_{\tau'}$ for each $\tau < \tau' < \omega_c$, $\{e_\tau : \tau < \omega_c\} \subset C$, $\{f_\tau : \tau < \omega_c\} \subset R \times R \times \ldots \times R \times D \times R \times \ldots R$ and $f(e_\tau, f_\tau) = t_\tau$ for each $\tau < \omega_c$, where $M = \{t_\tau : \tau < \omega_c\}$.

By Lemma 1 it follows that there exist sets A and B, with $A \subset \{e_\tau : \tau < \omega_c\}$ and $B = \{f_\tau : \tau < \omega_c\}$, such that f(AxB) is nonmeasurable. Furthermore A and B are both measurable since they are subsets of sets of measure zero.

Remark 2. One of the referees provided the following interesting application of Theorem 2. Let M_k be the set of all kxk matrices with real entries and $n=k^2$. If f(x,y) = xy, for $x,y,xy \in M_k$ (i.e. xy is the matrix multiplication of x and y), then there are Lebesgue measurable sets A and B of measure zero for which f(AxB) is nonmeasurable in the sense of Lebesgue.

Remark 3. The method of constructing sets C and D, in the proof of Theorem 1 is modelled after a construction in [4], on page 74.

Remark 4. Notice that the sets A and B constructed in the proof of Theorem 2 are both nowhere dense in R^n, and hence both have the Baire property and that f(AxB) does not have the Baire property ([3], page 24). This shows that Theorem 2 on page 257 in [1], i.e. the Baire set analogue of the Theorem of S. Kurepa, can be extended to functions f: $R^n \times R^n \to R^n$ satisfying the hypotheses of Theorem 2 in this paper.

Remark 5. The remarks about the continuum hypothesis, Martin's Axiom and (F) were brought to the authors attention by F. Galvin.

Remark 6. A. Abian and F. Galvin have pointed out that the n=1 case of Kurepa's Theorem goes back to Sierpiński (see [5]). Galvin has shown that the n=1 case of Kurepa's result implies the general case of Kurepa's result.

Remark 7. The author wishes to thank the referees for several remarks that helped improve the exposition in this paper.

Remark 8. We conclude by noting that in the proof of Theorem 2 it is not necessary to show that there exist "Cantor-like" sets C and D. Namely, since $\overline{N}_i(x_o,v)$ has n-dimensional Lebesque measure zero, it is sufficient to observe that

(the graph of g_{tj}) $\cap$ $(\overline{N}_i(x_o,v) \times D)$ clearly has cardinality c for each $t \in M$ if D is a "Cantor-like" subset of $\overline{N}(\hat{y},v)$ formed by taking out the middle n_o^{th} open intervals at each stage of its construction, where n_o is sufficiently large. The remainder of the proof of Theorem 2 goes through unchanged.

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