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THE LATTICE GENERATED BY DIFFERENTIABLE FUNCTIONS

The purpose of the present paper is to describe the lattice generated by the family of all differentiable functions. It answers a question posed by Z. Grande in [1].

Let us establish some of the terminology to be used. $\mathbb{R}$ denotes the real line. For every function $f:\mathbb{R} \to \mathbb{R}$ $N(f)$ denotes the set of all points at which f is not differentiable.

The symbol $\mathcal{D}$ stands for the family of all differentiable functions. The symbol $\mathcal{R}$ denotes the family of all continuous function $f:\mathbb{R} \to \mathbb{R}$ with the following properties:

(1) the set $N(f)$ is a finite union of discrete sets,

(2) for every $x \in \mathbb{R}$ the right-hand derivative $f'_+(x)$ and the left-hand derivative $f'_-(x)$ exist at x.

Observe that $\mathcal{D} \subset \mathcal{R}$.

A family $\mathcal{A}$ of real functions is a lattice iff $\max(f,g) \in \mathcal{A}$ and $\min(f,g) \in \mathcal{A}$ for $f,g \in \mathcal{A}$. If $\mathcal{B}$ is a family of real functions, then $\mathcal{L}(\mathcal{B})$ denotes the lattice generated by $\mathcal{B}$, i.e., the smallest lattice of functions containing $\mathcal{B}$.

The following question was posed in [1].

"Problem 7. What is the smallest lattice of functions containing all differentiable functions? Is it the family of all continuous functions differentiable at every point except perhaps at the points of a set which is a finite union of discrete sets?"

In this paper we shall prove that the lattice $\mathcal{L}(\mathcal{D})$ generated by the family of all differentiable functions is equal to the family $\mathcal{R}$.

Theorem 1. The family $\mathcal{R}$ is a lattice of functions.

Proof. Let $f,g \in \mathcal{R}$. We shall prove that $h = \max(f,g)$ belongs to $\mathcal{R}$. First, we shall verify that the set $M = N(h) \setminus (N(f) \cup N(g))$ is discrete and consequently, $N(h)$ is a finite union of discrete sets.

Notice that $h(x) = f(x) = g(x)$ for every $x \in M$. Let $x \in M$. Suppose that there exists a sequence $\{x_n\}_{n\in N}$ with $x_n \in M$ $(n = 1,2,\ldots,)$ and $\lim_{n\to\infty} x_n = x$. Then $f(x) = g(x)$ and $f(x_n) = g(x_n)$ for $n = 1,2,\ldots$. Since the functions f and g are differentiable,

$$f'(x) = \lim_{n\to\infty} \frac{f(x_n) - f(x)}{x_n - x} = \lim_{n\to\infty} \frac{g(x_n) - g(x)}{x_n - x} = g'(x).$$

Thus the function h is differentiable at x and $h'(x) = f'(x) = g'(x)$, contrary to $x \in M$. Hence for every $x \in M$ there exists a neighbourhood U of x such that $U \cap M = \{x\}$. Therefore the set M is discrete.

Let $x \in N(h)$. We prove that $h'_-(x)$ and $h'_+(x)$ exist. If $f(x) > g(x)$ $(f(x) < g(x))$, then there exists a neighborhood U of x with $h|_U = f|_U > g|_U$ $(f|_U < g|_U = h|_U)$. Hence $h'_-(x)$ and $h'_+(x)$ exist, $h'_-(x) = f'_-(x)$ $(h'_-(x) = g'_-(x))$ and $h'_+(x) = f'_+(x)$ $(h'_+(x) = g'_+(x))$.

Assume that $f(x) = g(x)$. There are two cases.

1. If $f'_-(x) = g'_-(x)$, then $h'_-(x) = f'_-(x)$.
2. If $f'_-(x) > g'_-(x)$, then

$$\lim_{z\to x^-} \frac{f(z) - f(x)}{z - x} > \lim_{z\to x^-} \frac{g(z) - g(x)}{z - x}.$$

Since $f(x) = g(x)$, we have

$$\lim_{z\to x^-} \frac{f(z) - g(z)}{z - x} > 0.$$

Since $z-x < 0$, there is an $a < x$ such that $f(z) - g(z) \le 0$ for every $z \in (a,x]$. Then $h|(a,x] = g|(a,x]$ and $h'_-(x) = g'_-(x)$. If $f'_-(x) < g'_-(x)$, the proof is analogous.

Similarly, we can prove that $h'_+(x)$ exists. (Notice additionally that $h'_-(x) = \min(f'_-(x), g'_-(x))$ and $h'_+(x) = \max(f'_+(x), g'_+(x))$.)

Thus, $\max(f,g) \in \mathcal{R}$ and similarly we can prove that $\min(f,g) \in \mathcal{R}$.

For $A \subset \mathbb{R}$ let $der(A)$ denote the set of all accumulation points of A which belong to A. Also, let $der^0(A) = A$ and $der^{k+1}(A) = der(der^k(A))$.

Lemma 1. (See [2], Lemma 2, pages 17-20.) For every $C \subset \mathbb{R}$ and for every $n = 1,2,\ldots$ the following statements are equivalent:

(i) C is a union of n discrete sets.

(ii) $\mathrm{der}^n(C) = \emptyset$.

Proof. If C is discrete, then $\mathrm{der}(C) = \emptyset$. Assume that $\mathrm{der}^n(C) = \emptyset$ whenever C is a union of n discrete sets.

Let C be a union of $n+1$ discrete sets $A_1, A_2, \ldots, A_{n+1}$. Observe that

$$\mathrm{der}^{n+1}(C) \subset \bigcup_{i=1}^{n+1} [A_i \cap \overline{\mathrm{der}^n(\bigcup_{j=1,j\neq i}^{n+1} A_j)}].$$

Hence $\mathrm{der}^{n+1}(C) = \emptyset$, and by induction we obtain that the implication (i) => (ii) holds for every $n \in \mathbb{N}$.

The implication (ii) => (i) is obvious since

$C = \bigcup_{k=0}^{n-1} (\mathrm{der}^k(C) \setminus \mathrm{der}^{k+1}(C))$ and the sets $\mathrm{der}^k(C) \setminus \mathrm{der}^{k+1}(C)$ are discrete.

Let us define the family $\mathcal{R}_n$ of all functions $f \in \mathcal{R}$ such that $\mathrm{der}^n(N(f)) = \emptyset$, for $n = 0,1,\ldots$. Notice that $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$.

Lemma 2. If $f \in \mathcal{R}_n$, then there exists functions $g_1, g_2, h_1, h_2 \in \mathcal{R}_{n-1}$ such that

$$f = \min(\max(g_1,g_2),\ \max(h_1,h_2)).$$

Proof. Let $f \in \mathcal{R}_n$ and $C = N(f) \setminus \mathrm{der}(N(f))$. Observe that the set C is discrete (and countable), $C \cap \mathrm{cl}\,\mathrm{der}(N(f)) = \emptyset$ and $\mathrm{der}(N(f))$ is a union of $n-1$ discrete sets.

Let $s : C \to \mathbb{N}$ be a one-to-one function with the following properties:

- if $f'_-(c) < f'_+(c)$, then $s(c)$ is even,

and

- if $f'_-(c) > f'_+(c)$, then $s(c)$ is odd.

Let $M = s(C)$ and $c_m = s^{-1}(m)$.

Let $\{(a_m,b_m)\}_{m\in M}$ be a sequence of pairwise disjoint intervals with $a \notin N(f)$, $b_m \notin N(f)$ and $(a_m,b_m) \cap N(f) = \{c_m\}$ for $m \in M$.

Let $p_{m,i} : [a_m,b_m] \to \mathbb{R}$ be differentiable functions with the following properties $(i = 1,2,\ m \in M)$:

(1) $p_{m,i}(a_m) = f(a_m)$ and $p_{m,i}(b_m) = f(b_m)$,

(2) $(p_{m,i})'_+ (a_m) = f'(a_m)$ and $(p_{m,i})'_- (b_m) = f'(b_m)$,

(3) $p_{m,1}|[a_m,c_m] = f|_{[a_m,c_m]}$ and $p_{m,2}|_{[c_m,b_m]} = f|_{[c_m,b_m]}$.

(4) $(p_{m,1})'_+ (c_m) = f'_- (c_m)$ and $(p_{m,2})'_- (c_m) = f'_+ (c_m)$,

(5) $p_{m,i}(x) \leq f(x)$ for $x \in [a_m,b_m]$, $i = 1,2$, $m \in M$, $m = 2k$,
and $p_{m,i}(x) \geq f(x)$ for $x \in [a_m,b_m]$, $i = 1,2$, $m \in M$, $m = 2k+1$,

(6) $|p_{m,i}(x) - f(x)| \leq \min((x-a_m)^2, (x-b_m)^2)$ for $x \in [a_m,b_m]$.

Let us define the functions $g_i,h_i (i = 1,2)$ such that $N(g_i) \subset der(N(f))$, $N(h_i) \subset der(N(f))$ and $g_i,h_i \in \mathcal{R}_{n-1}$ as follows:

$$g_i(x) = \begin{cases} f(x) & \text{for} \quad x \notin \bigcup_{m\in M} [a_m,b_m], \\ p_{m,i}(x) & \text{for} \quad x \in [a,b_m], \ m \in M \text{ and } m = 2k, \\ p_{m,1}(x) & \text{for} \quad x \in [a_m,b_m], \ m \in M \text{ and } m = 2k+1. \end{cases}$$

$$h_i(x) = \begin{cases} f(x) & \text{for} \quad x \notin \bigcup_{m\in M} [a_m,b_m], \\ p_{m,i}(x) & \text{for} \quad x \in [a_m,b_m], \ m \in M \text{ and } m = 2k, \\ p_{m,2}(x) & \text{for} \quad x \in [a_m,b_m], \ m \in M \text{ and } m = 2k+1. \end{cases}$$

Let us put $k_1 = \max(g_1,g_2)$, $k_2 = \max(h_1,h_2)$ and observe that $f = \min(k_2,k_2)$.

Theorem 2. We have the equality $\mathcal{L}(\mathcal{D}) = \mathcal{R}$.

Proof. Since $\mathcal{D} \subset \mathcal{R}$ and the family $\mathcal{R}$ is a lattice, we have $\mathcal{L}(\mathcal{D}) \subset \mathcal{R}$. By Lemma 2 it follows that $\mathcal{R}_n \subset \mathcal{L}(\mathcal{D})$ for each $n \in N$. Hence $\mathcal{R} \subset \mathcal{L}(\mathcal{D})$ and the equality $\mathcal{L}(\mathcal{D}) = \mathcal{R}$ is proved.

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Remark. Prof. Z. Grande has remarked that in his proof of Theorem 7 [1] the statement

$$(5) \qquad |h_n(x)| \le \max((x-a_n + r_n)^2, \ (x-a_n - r_n)^2)$$

must be added.

References

1. Z. Grande, Some problems in differentiation theory, Real Analysis Exchange, Vol. 10, No. 2 (1984-85), pp. 334-343.

2. T. Natkaniec, Sets of points of continuity and semicontinuity of real functions (in Polish), Ph.D. Dissertation, Łodz, (1984).

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