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Perfect level sets in many directions

The so-called locally recurrent functions are defined by the following property: for every $c \in R$ the set $\{x : f(x) = c\}$ is perfect. (We include the empty set among the perfect sets.) The existence of a nonconstant continuous localy recurrent function is not obvious, but there are several known examples. (See e.g. Bush (1962, [2]).) In the present paper we construct a continuous function f on $[0,1]$ such that the functions $f - \lambda \mathrm{id}$ are locally recurrent for every $\lambda \in \Lambda$, where $\Lambda \subset \mathbb{R}$ is a given countable set. (The function id is defined by $\mathrm{id}(x) = x$.) This construction cannot be improved to Λ being uncountable because of the results of Bruckner and Garg (1977, [1]) concerning the level sets of arbitrary continuous functions. (Gillis (1939, [3]) claimed that one can take $\Lambda = \mathbb{R}$, but this is a mistake.) Further, we show that every continuous function on $[0,1]$ can be expressed as the sum of two locally recurrent functions.

Definition. A function u on $[0,1]$ is termed admissible if there is a finite set $A_u \subset (0,1)$ such that

(1) if $I \subset [0,1]\setminus A_u$ is an interval, then u is linear on I,

(2) $u(x) < \liminf_{y \to x} u(y)$ for every $x \in A_u$.

Lemma. Let $s, -t$ be admissible functions on $[0,1]$, $t \leq s$ on $[0,1]$ and $t < s$ except on a finite set. Let $\varepsilon > 0$ be given. Then there are admissible functions $s^*, -t^*$ on $[0,1]$ such that

(1) $t \leq t^* \leq s^* \leq s$ on $[0,1]$,

(2) $t^* < s^*$ except on a finite set,

(3) $s^* - t^* < \varepsilon$ on $[0,1]$,

(4) if f is a continuous function on $[0,1]$, $t^* \leq f \leq s^*$, then for each $x \in [0,1]$ there is $y \in [0,1]$ such that $0 < |x - y| < \varepsilon$ and $f(x) = f(y)$.

Proof. We can easily find a continuous function g on $[0,1]$ such that $t \leq g \leq s$ on $[0,1]$ and $t < g < s$ wherever $t < s$. Using the uniform continuity of g on $[0,1]$ we find a partition

$$0 = z_0 < z_1 < \dots < z_p = 1$$

of the interval $[0,1]$ such that for every $j = 1,\dots,p$ and $I = [z_{j-1},z_j]$ we have $z_j - z_{j-1} < \varepsilon$, and one of the following situations happens:

(1) $(A_s \cap I) \cup (A_{-t} \cap I) \subset \{z\}$ for some $z \in (z_{j-1},z_j)$, there are $\alpha,\beta \in \mathbb{R}$ such that $\beta - \alpha < \varepsilon$, $t \leq \alpha < g < \beta \leq s$ on $I\setminus\{z\}$,

(2) $A_s \cap I = \{z\}$ for some $z \in (z_{j-1},z_j)$, $s(z) = t(z)$, t is linear but nonconstant on I, there are $\alpha,\beta \in \mathbb{R}$ such that $\beta - \alpha < \varepsilon$, $\inf_I t < \alpha < g < \beta \leq s$ on I,

(3) $A_{-t} \cap I = \{z\}$ for some $z \in (z_{j-1},z_j)$, $s(z) = t(z)$, s is linear but nonconstant on I, there are $\alpha,\beta \in \mathbb{R}$ such that $\beta - \alpha < \varepsilon$, $t \leq \alpha < g < \beta < \sup_I s$ on I.

Fix $j \in \{1,\dots,p\}$ and let $I = [z_{j-1},z_j]$. If (1) holds on I, choose points $y_i \in (z_{j-1},z_j)\setminus\{z\}$, $i = 1,2,3,4$, $y_1 < y_2 < y_3 < y_4$, and put

$$s^* = \begin{cases} \alpha & \text{on } \{y_1,y_3\} \\ g & \text{on } \{z_{j-1},z,z_j\}\setminus\{0,1\} \\ \beta & \text{elsewhere on } I, \end{cases}$$

$$t^* = \begin{cases} \beta & \text{on } \{y_2,y_4\} \\ g & \text{on } \{z_{j-1},z,z_j\}\setminus\{0,1\} \\ \alpha & \text{elsewhere on } I. \end{cases}$$

If (2) holds on I, choose points $y_i \in (z_{j-1},z_j)$, $i = 1,2,3,4$, such that $t(y_i) < \alpha$ and $y_1 < y_2 < y_3 < y_4$. Further denote by y_0 the point of I satisfiying $t(y_0) = \alpha$. Define s^* as in the previous case and put

$$t^* = \begin{cases} \beta & \text{on } \{y_2,y_4,y_0\} \\ g & \text{on } \{z_{j-1},z,z_j\}\setminus\{0,1\} \\ \max(t,\alpha) & \text{elsewhere on } I. \end{cases}$$

If (3) holds on I, we proceed symmetrically as in (2). It is easy to see that we have constructed admissible functions s^*, $-t^*$ on $[0,1]$, $t \leq t^* \leq s^* \leq s$, $s^* - t^* < \varepsilon$ and $t^* < s^*$ except on a finite set. Choose a continuous function f on $[0,1]$, $t^* \leq f \leq s^*$, and $x \in [0,1]$. Find $j \in \{1,\dots,p\}$ such that $x \in [z_{j-1}, z_j]$. Let α, β, y_i be as in the construction. Let J be one of the intervals $[y_1,y_2]$, $[y_3,y_4]$ such that $x \notin J$. Then we have

$$\inf_J f = \alpha \leq f(x) \leq \beta = \sup_J f .$$

By the Darboux property of continuous functions there is $y \in J$ with $f(y) = f(x)$. Of course, $0 < |x - y| \leq |z_j - z_{j-1}| < \varepsilon$.

Theorem. Let $\Lambda \subset \mathbb{R}$ be a countable set. Then there is a continuous function f on $[0,1]$ such that for every $\lambda \in \Lambda$ and $c \in \mathbb{R}$ the set

$$\{x : f(x) = \lambda x + c\}$$

is perfect.

Proof. Let $\{\lambda_n\}$ be a sequence of reals such that every $\lambda \in \Lambda$ equals λ_n for infinitely many indices n. By means of induction we construct sequences $\{s_n\}$, $\{-t_n\}$ of admissible functions on $[0,1]$ such that

(1) $t_1 \leq t_2 \leq \dots \leq s_2 \leq s_1$,
(2) $|s_n - t_n| < 2^{-n}$, $t_n < s_n$ except on a finite set,
(3) if h is a continuous function on $[0,1]$ with $t_n \leq h \leq s_n$, then for every

$x \in [0,1]$ there is $y \in [0,1]$ such that

$$0 < |x - y| < 2^{-n} \text{ and } h(x) - \lambda_n x = h(y) - \lambda_n y$$

Namely, apply the Lemma to $s = s_{n-1} - \lambda_n \mathrm{id}$, $t = t_{n-1} - \lambda_n \mathrm{id}$ (taking $s_0 = 1$, $t_0 = 0$ for the first step $n = 1$) and put $s_n = s^* + \lambda_n \mathrm{id}$, $t_n = t^* + \lambda_n \mathrm{id}$.

It is easy to see that the function

$$f = \lim_n s_n = \lim_n t_n$$

has the required properties.

Remark. Let g be a continuous function on $[0,1]$, and let

$$B = \{g(y) : g \text{ has a strict local extremum at } y\} .$$

The set B is countable. Let us construct a continuous function f on $[\min g - 1, \max g + 1]$ such that both the functions $f + id$, $f - id$ are locally recurrent. By a slightly more careful treatment of the construction, we can achieve that every $x \in B$ is a bilateral point of accumulation of the sets

$$\{y : f(y) - y = f(x) - x\} ,$$

$$\{y : f(y) + y = f(y) + x\} .$$

Then the functions $\frac{1}{2}(g - f \cdot g)$, $\frac{1}{2}(g + f \cdot g)$ are locally recurrent and their sum equals g.

References

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