TOPICAL SURVEY Real Analysis Exchange Vol. 12 (1986-87)

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A CONCEPT OF DIFFERENTIAL BASED ON VARIATIONAL EQUIVALENCE UNDER GENERALIZED RIEMANN INTEGRATION

For appropriate types of integral the variational equivalence $S \sim T$ of two objects of integration S, T is the relation $\int |S-T| = 0$. Kolmogorov [11] introduced such a notion, aptly called "differential equivalence," for set functions. He discussed its basic properties and even noted the differential invariance of Lipschitz functions. Variational equivalence has been used in the development of the generalized Riemann integral [6], [7], [8]. It is essential for the definition of the variational integral. But we contend it has a more important role to play. If $S \sim T$ and S is integrable then so is T and moreover $\int S = \int T$. Thus the ultimate object of integration in not S itself but the equivalence class $\sigma = [S]$ to which S belongs. Our contention here is that these equivalence classes provide a viable mathematical formulation for a concept of differential. Differentials defined in this way greatly facilitate the study of the integral and afford easy access to its applications. We gain a rigorous foundation for a calculus of differentials that includes differentials of discontinuous functions.

In this survey we explore the feasibility of integrational definition of differential by applying it to the exposition of a specific type of integral. We use a modification of Kurzweil's generalized Riemann integral [8]. Where Kurzweil allows the tag for a cell to be any point in the cell we demand that the tag be a vertex of the cell. The differentials induced by this integral have many desirable properities. A suitable subclass of them conforms to the classical formulas of differential calculus. Our differentials yield elegant formulations for arc length in an arbitrary norm. They offer some new concepts that should prove useful for analysis.

Hopefully this survey will motivate analysts to study differentials induced by this and other types of integrals [9]. Such studies could yield new perspectives on differentials in their various manifestations.

We shall define m-differentials on an n-cell K (a product of n closed intervals) and more generally on an n-figure (a finite union of n-cells). The m-differentials on K form a Riesz (lattice-ordered, linear) space on which all 1-functions on K act as multipliers. If $\|\ \|$ is any norm on $\mathbb{R}^m$ and σ is an m-differential on K then $\|\sigma\|$ is a 1-differential on K. Every m-differential σ on K has a lower and upper integral with values in $[-\infty, \infty]^m$. σ is integrable whenever these are equal and finite. Every m-function $x = (x_1,\ldots,x_m)$ on K has an integrable m-differential $dx = (dx_1,\ldots,dx_m)$ with $\int_K dx = \Delta x(K)$ where Δ is the operator product of the partial difference operators in each coordinate across K. The differential $|dx| = (|dx_1|,\ldots,|dx_m|)$ is integrable whenever x is of bounded variation, that is, whenever the 1-differential $\|dx\|_1 = |dx_1| + \ldots + |dx_m|$ is integrable on K. For x of unbounded variation $\int_K \|dx\|_1 = \infty$. Every integrable m-differential on K is the differential dx of some m-function x on K. Under mild restrictions which always hold in classical applications we get the chain rule formulas and the existence of various products of differentials.

For the generalized Riemann integral [8], [12], and especially [17] are helpful. But none of these is essential here. An exposition of 1-differentials on 1-cells along the lines developed here is given in [16]. An extensive bibliography for the generalized Riemann integral can be found in [19].

For basic facts about Riesz spaces see [5]. We use only standard analysis here. A nonstandard approach to the generalized Riemann integral can be found in [1].

1. PRELIMINARY DEFINITIONS. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{R}$ the set of all real numbers, and $\mathbb{R}_+$ the set of all $t > 0$ in $\mathbb{R}$. An *m-function* is any mapping into $\mathbb{R}^m$. For $a = (a_1,\ldots,a_n)$ and $b = (b_1,\ldots,b_n)$ in $\mathbb{R}^n$ define *$a \leq b$ ($a < b$)* to be $a_i \leq b_i$ (respectively $a_i < b_i$) for $i = 1,\ldots,n$. Given $a < b$ define the *n-cell* $[a,b]$ to be the set of all t in $\mathbb{R}^n$ such that $a \leq t \leq b$. Since $[a,b]$ is just the cartesian product of the 1-cells $[a_i, b_i]$, t is interior to $[a,b]$ if and only if $a < t < b$. A point t in $\mathbb{R}^n$ is a *vertex* of $[a,b]$ if t_i is an endpoint of $[a_i,b_i]$ for $i = 1,\ldots,n$. A *tagged* n-cell (I,t) is an n-cell I with selected vertex t. An *n-figure* F is a nonvoid union of finitely many n-cells in $\mathbb{R}^n$. Two n-figures *overlap* if their intersection contains an n-cell. A finite set of n-cells *partitions* (is a *partition* of) their union F if no two of them overlap. A *division* $\mathcal{F}$ of an n-figure F is a finite set of tagged n-cells which partition F. A *gauge* on F is a function δ on F into $\mathbb{R}_+$. A tagged n-cell (I,t) is *δ-fine* if I is contained in the Euclidean ball of radius $\delta(t)$ about t. A *δ-division* is a division whose members are δ-fine. For any gauge δ on an n-cell K the existence of a δ-division of K can be proved by induction on n using a Heine-Borel argument [17]. Thus, since every n-figure F can be partitioned into n-cells, every δ-division of an n-figure contained in F can be extended to a δ-division of F.

An *m-summant* S on an n-figure F is an m-function $S(I,t)$ on the set of all tagged n-cells (I,t) in F. For such S each division $\mathcal{F}$ of F yields a *Riemann sum* $\Sigma(S,\mathcal{F})$, the sum of $S(I,t)$ over all (I,t) in $\mathcal{F}$

S is <u>integrable</u> with <u>integral</u> $\int S = c$ in $\mathbb{R}^m$ if given ε in $\mathbb{R}_+$ there exists a gauge δ on F such that in the Euclidean norm $||c-\Sigma(S,\mathscr{F})|| < \varepsilon$ for every δ-division $\mathscr{F}$ of F. A necessary and sufficient condition for S to be integrable is the Cauchy criterion: given ε in $\mathbb{R}_+$ there exists a gauge δ on F such that $||\Sigma(S,\mathscr{F}_1) - \Sigma(S,\mathscr{F}_2)|| < \varepsilon$ for all δ-divisions $\mathscr{F}_1$ and $\mathscr{F}_2$ of F. The function space $\mathbf{S}_m$ of all m-summants on F is a Riesz space. The integrable m-summants form a linear subspace $\mathbf{I}_m$ of $\mathbf{S}_m$ on which the integral acts as a positive linear m-function. But $\mathbf{I}_m$ is not a Riesz space. It fails to be a lattice because integrability of $S = (S_1, \ldots, S_m)$ does not imply integrability of $|S| = (|S_1|,\ldots,|S_m|)$ although it does imply (See THEOREM 4) that $\int|S|$ exists in $[0,\infty]^m$. S is <u>absolutely integrable</u> if both S and $|S|$ are integrable. Clearly S is integrable if and only if all its component 1-summants $S_1,\ldots,S_m$ are.

For S an m-summant on F define the lower and upper integrals to be the lower and upper limits in $[-\infty,\infty]^m$ of the Riemann sums of S. Explicitly for each gauge δ on F define $\underline{\Sigma}(S,\delta)$ to be the infimum in $[-\infty,\infty]^m$, $\overline{\Sigma}(S,\delta)$ the supremum, of all sums $\Sigma(S,\mathscr{F})$ with $\mathscr{F}$ any δ-division of F. Define the <u>lower integral</u> $\underline{\int} S = \sup_\delta \underline{\Sigma}(S,\delta)$ and the <u>upper integral</u> $\overline{\int} S = \inf_\delta \overline{\Sigma}(S,\delta)$ where δ runs through all gauges on F. For $S = (S_1,\ldots,S_m)$ we clearly have $\underline{\int} S = (\underline{\int} S_1,\ldots,\underline{\int} S_m)$ and $\overline{\int} S = (\overline{\int} S_1,\ldots,\overline{\int} S_m)$. S is integrable if and only if its lower and upper integrals are equal and finite. Moreover, $\int S = \underline{\int} S = \overline{\int} S$ for S integrable. By extension we use this to define $\int S$ in $[-\infty,\infty]^m$ whenever $\underline{\int} S = \overline{\int} S$.

A <u>cell summant</u> is a summant $S(I,t) = S(I)$ whose values are independent of the tag t. Similarly a <u>tag summant</u> $T(I,t) = T(t)$ depends only on the tag t.

2. DIFFERENTIALS. Hereafter F will always be an n-figure. In the Riesz

space $\mathcal{S}_m$ of all m-summants on F those S for which $\int|S| = 0$ form a <u>Riesz ideal</u> $\mathbf{Z}_m$. That is, $\mathbf{Z}_m$ is a linear subspace of $\mathcal{S}_m$ which is <u>solid</u>: If $S \in \mathcal{S}_m$, $T \in \mathbf{Z}_m$, and $|S| < |T|$ then $S \in \mathbf{Z}_m$. Thus $\mathbf{D}_m = \mathcal{S}_m/\mathbf{Z}_m$ is a Riesz space with the linear and lattice operations transferred homomorphically from $\mathcal{S}_m$ to $\mathbf{D}_m$. We define an <u>m-differential</u> σ on F to be any element of $\mathbf{D}_m$. Explicitly σ is an equivalence class $[S]$ of m-summants on F under the equivalence <u>$S \sim T$</u> defined by $\int|S - T| = 0$. $S \sim T$ if and only if $S_i \sim T_i$ for $i = 1,\dots,m$. So $\sigma = [S]$ has 1-differential components $\sigma_i = [S_i]$ for $S = (S_1,\dots,S_m)$. We express this as $\sigma = (\sigma_1,\dots,\sigma_m)$. For m-differentials $\rho = [R]$ and $\sigma = [S]$ on F the homomorphism gives $\rho+\sigma = [R + S]$, $c\sigma = [cS]$ for any scalar c, $|\sigma| = [|S|]$, $\rho\wedge\sigma = [R\wedge S]$, $\rho\vee\sigma = [R\vee S]$, $\sigma^+ = [S^+]$, and $\sigma^- = [S^-]$. It is useful to transfer the differential ordering $\rho < \sigma$ defined by $(\rho - \sigma)^+ = 0$ to representative summants. So define $R \lesssim S$ to be $\int(R - S)^+ = 0$. Then $R \sim S$ if and only if both $R \lesssim S$ and $S \lesssim R$. It is easy to see that $R \lesssim S$ implies $\underline{\int} R < \underline{\int} S$ and $\overline{\int} R < \overline{\int} S$. So $R \sim S$ implies $\underline{\int} R = \underline{\int} S$ and $\overline{\int} R = \overline{\int} S$. Thus we can effectively define the lower and upper integrals of any differential $\sigma = [S]$ by $\underline{\int}\sigma = \underline{\int} S$ and $\overline{\int}\sigma = \overline{\int} S$. Define $\int\sigma = \underline{\int}\sigma = \overline{\int}\sigma$ whenever the lower and upper integrals are equal. Call σ <u>integrable</u> (<u>absolutely integrable</u>) whenever S is so, respectively. Define $\mathfrak{m}(\sigma) = \|\overline{\int}|\sigma|\|_1 = \Sigma_{i=1}^{m} \overline{\int}|\sigma_i|$ for every m-differential $\sigma = (\sigma_1,\dots,\sigma_m)$ on F. $\mathfrak{m}$ has all the properties of a Riesz norm (a norm such that $\mathfrak{m}(\rho) < \mathfrak{m}(\sigma)$ whenever $|\rho| < |\sigma|$) on $\mathbf{D}_m$ except that it is <u>improper</u>: $\mathfrak{m}(\sigma) = \infty$ for some σ. Indeed, $\mathbf{D}_m$ being nonarchimedean admits no proper Riesz norm. Scalar multipliers have discontinuities.

Let Z be mapping of $\mathcal{S}_m$ into $\mathcal{S}_k$ for which there exists c in $\mathbb{R}_+$ such that $\|(Z(S) - Z(T))\,(I,t)\|_1 < c\|(S - T)(I,t)\|_1$ for all S, T in $\mathcal{S}_m$ and all tagged n-cells (I,t) in F. Under this Lipschitz condition Z transfers homomorphically to a mapping of $\mathbb{D}_m$ into $\mathbb{D}_k$ effectively defined by $Z(\sigma) = [Z(S)]$ for $\sigma = [S]$. Later it will be clear that this is effective even if c is a positive 1-function on F.

Every Lipschitz k-function f on $\mathbb{R}_m$ induces a transferable mapping Z defined by $Z(S)(I,t) = f(S(I,t))$. We can apply this to any norm $f(r) = ||r||$ on $\mathbb{R}^m$ to get a 1-differential $||\sigma|| = [||S||]$ for every m-differential $\sigma = [S]$. Clearly $||\sigma|| = 0$ if and only if $\sigma = 0$. Also $||\sigma|| \geq 0$, $||\sigma + \tau|| \leq ||\sigma|| + ||\tau||$, and $||c\sigma|| = |c|\ ||\sigma||$ for m-differentials σ,τ on F and scalar c.

3. INTEGRABLE DIFFERENTIALS. A coordinate hyperplane H in $\mathbb{R}^n$ cuts an n-cell K if H intersects the interior of K (thereby partitioning K into two abutting n-cells). Our first theorem exploits the restriction of tags to the vertices of cells.

THEOREM 1. Given a partition $\mathcal{P}$ of F there exists a gauge δ on F such that every δ-division of F refines $\mathcal{P}$.

PF. Let $H_1,\ldots,H_k$ be all the coordinate hyperplanes which pass through any vertices of n-cells belonging to $\mathcal{P}$. An n-cell in F which is not cut by any H_j must be contained in some member of $\mathcal{P}$. Take δ on F fine enough so that $\delta(t)$ is less than the distance from t to each H_j that does not contain t. Consider any δ-fine (I,t) in F. Since t is a vertex of I no coordinate hyperplane through t cuts I. By our choice of δ no H_j which avoids t can cut I. So no H_j cuts I.

Hereafter we shall use $\int_F$ in place of $\int$ wherever the figure over which we are integrating may be amibiguous.

THEOREM 2. Let σ be an m-differential on the union C of two non-overlapping n-figures A,B. Then $\underline{\int}_C \sigma = \underline{\int}_A \sigma + \underline{\int}_B \sigma$ and $\overline{\int}_C \sigma = \overline{\int}_A \sigma + \overline{\int}_B \sigma$ ignoring the indeterminate form $\infty - \infty$. If σ is integrable on both A and B then σ is integrable on C and $\int_C \sigma = \int_A \sigma + \int_B \sigma$.

PF. Let $\mathcal{A}_0$ partition A and $\mathcal{B}_0$ partition B. Then $\mathcal{C}_0 = \mathcal{A}_0 \cup \mathcal{B}_0$ partitions C. By THEOREM 1 there exists a gauge δ on C such that every δ-division $\mathcal{C}$ of C refines $\mathcal{C}_0$. Thus $\mathcal{C}$ is the union of δ-divisions $\mathcal{A}$ of A and $\mathcal{B}$ of B. So $\Sigma(S,\mathcal{C}) = \Sigma(S,\mathcal{A}) + \Sigma(S,\mathcal{B})$ for any summant S. Since δ-fine $\mathcal{A},\mathcal{B}$ may be chosen independently to form such $\mathcal{C}$ we have THEOREM 2.

THEOREM 3. Let σ be an integrable m-differential on F. Let $S \in \sigma$. Then S is uniformly integrable on all n-figures E contained in F. Specifically if δ is a gauge on F such that

(1) $||\Sigma(S,\mathcal{F}) - \int_F \sigma|| < \varepsilon$ for every δ-division $\mathcal{F}$ of F

then for every n-figure E contained in F

(2) $||\Sigma(S,\mathcal{E}) - \int_E \sigma|| < \varepsilon$ for every δ-division $\mathcal{E}$ of E.

PF. Let the n-figure D be the closure of F\E. Given ε in $\mathbb{R}_+$ take a gauge δ on F so that (1) holds. Take any δ-division $\mathcal{D}$ of D. Given δ-divisions $\mathcal{E}_1, \mathcal{E}_2$ of E let $\mathcal{F}_i = \mathcal{D} \cup \mathcal{E}_i$ for $i = 1,2$. Each $\mathcal{F}_i$ is a δ-division of F. So $||\Sigma(S, \mathcal{E}_1) - \Sigma(S, \mathcal{E}_2)|| = ||\Sigma(S,\mathcal{F}_1) - \Sigma(S,\mathcal{F}_2)|| < 2\varepsilon$ by (1). The Cauchy criterion for integrability of S on E is thus satisfied. So σ is integrable on E. For $\mathcal{E} = \mathcal{E}_1$ and $\mathcal{F} = \mathcal{F}_1$ THEOREM 2 and (1) imply $||\Sigma(S,\mathcal{E}) - \int_E \sigma|| = ||\Sigma(S,\mathcal{F}) - \Sigma(S,\mathcal{D}) + \int_D \sigma - \int_F \sigma|| <$ $||\Sigma(S,\mathcal{F}) - \int_F \sigma|| + ||\int_D \sigma - \Sigma(S,\mathcal{D})|| < \varepsilon + ||\int_D \sigma - \Sigma(S,\mathcal{D})||$. Taking $\mathcal{D}$ fine enough we can force the last norm towards 0 giving (2).

Call a summant S on F <u>additive</u> if S is a cell summant such that $S(K) = \Sigma(S,\mathcal{P})$ for every n-cell K in F and partition $\mathcal{P}$ of K. It suffices for this to hold for all two-member partitions since every partition has a refinement formed by finitely many cuts with coordinate hyperplanes.

THEOREM 4. Let σ be an integrable m-differential on F. Define the additive summant $\hat{S}$ on F by

(3) $\hat{S}(I) = \int_I \sigma$ for every n-cell I contained in F.

Then $\hat{S} \in \sigma$. For $\| \ \|$ any norm on $\mathbb{R}^m$, $\int_F \|\sigma\|$ exists in $[0,\infty]$ and

(4) $\int_F \|\sigma\| = \sup_{\mathcal{P}} \Sigma(\|\hat{S}\|,\mathcal{P})$ over n-cell partitions $\mathcal{P}$ of F.

Similarly $\int_F |\sigma| = \sup_{\mathcal{P}} \Sigma(|\hat{S}|, \mathcal{P})$ in $[0,\infty]^m$.

PF. Existence and additivity of $\hat{S}$ follow from THEOREM 3 and THEOREM 2. Let $S \in \sigma$. Given ε in $\mathbb{R}_+$ take a gauge δ on F such that (1) holds for $\| \ \|_1$. Given any δ-division $\mathcal{F}$ of F and $1 \le i \le m$ let $\mathcal{E}_i$ consist of all members of $\mathcal{F}$ at which $S_i > \hat{S}_i$. Let E_i be the union of the cells from $\mathcal{E}_i$. For each i, $\Sigma((S_i - \hat{S}_i)^+, \mathcal{F}) = \Sigma(S_i - \hat{S}_i, \mathcal{E}_i) = \Sigma(S_i, \mathcal{E}_i) - \int_{E_i}\sigma_i \le \|\Sigma(S, \mathcal{E}_i) - \int_{E_i}\sigma\|_1 < \varepsilon$ by THEOREM 2 and THEOREM 3. Summing over i we get $\|\Sigma((S-\hat{S})^+, \mathcal{F})\|_1 < m\varepsilon$. So $\|\Sigma(|S-\hat{S}|,\mathcal{F})\|_1 < 2m\varepsilon$ for every δ-division $\mathcal{F}$ of F. Hence $\hat{S} \sim S$. So $\hat{S} \in \sigma$. Since $\|\hat{S}\|$ is subadditive $\Sigma(\|\hat{S}\|,\mathcal{P}) \le \Sigma(\|\hat{S}\|, \mathcal{Q})$ for $\mathcal{Q}$ a refinement of $\mathcal{P}$. Given a partition $\mathcal{P}$ of F choose δ by THEOREM 1. Then $\Sigma(\|\hat{S}\|,\mathcal{P}) \le \Sigma(\|\hat{S}\|, \mathcal{F})$ for every δ-division $\mathcal{F}$ of F. This gives (4). Apply (4) to each component σ_i of σ to prove the final statement in THEOREM 4. (The statement that $\hat{S} \in \sigma$ is a differential formulation of Henstock's Lemma.)

Using additive summants one can easily see that the integrable m-differentials on F form a complete topological group under addition and the improper norm $\mathfrak{n}(\sigma) = \int_F ||\sigma||_1$.

Let us characterize the ordering for integrable differentials.

<u>THEOREM 5.</u> Let σ be an integrable m-differential on F. Then $\sigma > 0$ if and only if $\int_I \sigma > 0$ for every n-cell I in F. So $\sigma = 0$ if and only if $\int_I \sigma = 0$ for every n-cell I in F.

PF. Given $\sigma > 0$ let $S \varepsilon \sigma$. Then $S^+ \varepsilon \sigma^+ = \sigma$. So $\int_I \sigma = \int_I S^+ > 0$. Conversely if $\int_I \sigma > 0$ for all I then $\hat{S} > 0$ by (3). $\hat{S} \varepsilon \sigma$ by THEOREM 4. So $\hat{S} = \hat{S}^+ \varepsilon \sigma^+$. Hence $\sigma = \sigma^+ > 0$.

A <u>regular closed</u> set A is the closure $\overline{U}$ of an open set U, in particular the closure of the interior $\overset{\circ}{A}$ of A. The regular closed subsets of a regular closed set C form a boolean algebra $\mathcal{R}(C)$ with lattice operations $A \vee B = A \cup B$, $A \wedge B = \overline{(A \cap B)^{\circ}}$, and $A^{-} = \overline{C \setminus A}$. The n-figures contained in an n-cell $K = [a,b]$ form a boolean subalgebra $\mathcal{F}(K)$ of $\mathcal{R}(K)$. Let us review some facts [15].

Let $(-\infty,t]$ consist of all s in $\mathbb{R}^n$ with $s < t$. The set of all $(-\infty,t]$ with t in K forms a meet-semilattice $\mathcal{B}(K)$ which is a basis for the boolean subalgebra $\mathcal{A}(K)$ of $\mathcal{R}(-\infty,b]$ that it generates. That is, every m-function $S(-\infty,t]$ on $\mathcal{B}(K)$ has a unique extension to an additive m-function S on $\mathcal{A}(K)$, $S(A \cup B) = S(A) + S(B)$ for nonoverlapping A,B in $\mathcal{A}(K)$. Moreover $\mathcal{F}(K)$ is contained in $\mathcal{A}(K)$. For $I = [q,r]$ an n-cell in K the extension is prescribed by the inclusion-exclusion formula

$$S(I) = \sum_{t \varepsilon V} (-1)^{M(I,t)} S(-\infty,t] \tag{5}$$

where V is the set of all 2^n vertices of I and $M(I,t)$ is the number of coordinates of t for which $t_i = q_i$. Given an m-function x on K we can set $S(-\infty,t] = x(t)$ in (5) to define an additive m-summant Δx on K by

$$\Delta x(I) = \sum_{t \varepsilon V} (-1)^{M((I,t)} x(t). \tag{6}$$

Given h in $\mathbb{R}^n$ let $h(i)$ be the point in $\mathbb{R}^n$ with i-th coordinate h_i and all other coordinates 0. Define the shift operator E_i by $(E_i x)(t) = x(t + h(i))$. Let $\Delta_i = E_i - 1$ where 1 is the identity operator. Since E_i, ..., E_n commute the sum in (6) is the expansion of $\Delta_1 \dots \Delta_n = (E_1 - 1) \dots (E_n - 1)$ acting on $x(t)$ at $t=q$ with $h = r-q$. (This formula gives an alternate proof of the additivity of Δx.)

Define the m-differential dx on K by

$$dx = [\Delta x]. \tag{7}$$

The operator Δ in (6) maps m-functions linearly into m-summants. So d maps m-functions linearly into m-differentials. d maps constant functions into 0.

Hereafter we may revert to the notation $\int_a^b$ for $\int_{[a,b]}$. We also define $\int_a^b \sigma = 0$ for σ any differential on an n-cell containing a,b if [a,b] is <u>degenerate</u>, $a \le b$ but $a \not< b$.

<u>THEOREM 6.</u> An m-differential σ on F is integrable if and only if $\sigma = dx$ for some m-function x on F. $\Delta x(I) = \int_I dx$ for every m-function x on F and every n-cell I contained in F.

PF. Let σ be integrable on F. By THEOREM 2 and THEOREM 3 we need only consider the case where F is an n-cell [a,b]. Define x by

$$x(t) = \int_a^t \sigma \quad \text{for} \quad a \le t \le b \tag{8}$$

which exists by THEOREM 3. $x(t) = \hat{S}[a,t]$ for $a \leq t \leq b$ by (8), (3). So $\Delta x = \hat{S}$ by (5),(6). Hence $dx = \sigma$ by THEOREM 4. The rest of THEOREM 6 follows trivially from (7) since Δx is additive.

Let x be an m-function on an n-cell $K = [a,b]$ with $|dx|$ integrable. By THEOREM 6 there exists an m-function v on K with $dv = |dx|$, namely $v(t) = \int_a^t |dx|$. Let $y = (v+x)/2$ and $z = (v-x)/2$ to get the Jordan decomposition $x = y-z$ with $v = y+z$, $dy = (dx)^+$ and $dz = (dx)^-$.

Using THEOREM 6 and (6) it is an easy exercise to show that given an integrable m-differential σ on an n-cell K, a point c in K, and an m-function w on the set L of all t in K such that $t_i = c_i$ for some i, there is a unique m-function x on K such that $dx = \sigma$ and $x(t) = w(t)$ for all t in L.

Every 1-function z on F defines a multiplication operator on $\mathcal{S}_m$ by $(zS)(I,t) = z(t)S(I,t)$ for each m-summant S on F. If z is bounded this is a Lipschitz operator, so $z\sigma = [zS]$ is defined for $\sigma = [S]$. This definition turns out to be effective even if z is unbounded. Our next section will show this.

4. MONOTONE CONVERGENCE. For THEOREM 7 we need two lemmas.

<u>LEMMA A.</u> Let $S \geq 0$ be a 1-summant on F. Let α,β be gauges on F such that $\alpha(t) \leq \beta(t)$ for all t where $S(I,t) > 0$ for some n-cell I tagged by t. Then $\overline{\Sigma}(S,\alpha) \leq \overline{\Sigma}(S,\beta)$.

PF. Given an α-division $\mathcal{F}$ of F let $\mathcal{F}_0$ consist of all members of $\mathcal{F}$ for which $S = 0$, and $\mathcal{F}_+$ those for which $S > 0$. Let G be the figure with division $\mathcal{F}_0$. Take any β-division $\mathcal{G}$ of G and let $\mathcal{F}' = \mathcal{F}_+ \cup \mathcal{G}$. Then $\mathcal{F}'$ is a β-division of F and $\Sigma(S,\mathcal{F}) = \Sigma(S,\mathcal{F}_+) \leq \Sigma(S,\mathcal{F}_+) + \Sigma(S,\mathcal{G}) = \Sigma(S,\mathcal{F}')$. So $\Sigma(S,\mathcal{F}) \leq \overline{\Sigma}(S,\beta)$ for every α-division $\mathcal{F}$ of F. Hence

$\overline{\Sigma}(S,\alpha) \leq \overline{\Sigma}(S,\beta)$.

LEMMA B. Let $T, T_1, T_2, \ldots \geq 0$ be 1-summants on F such that given $q < 1$ in $\mathbb{R}_+$ there exists a function r on F into $\mathbb{N}$ for which $qT(I,t) \leq \Sigma_{k=1}^{r(t)} T_k(I,t)$ for all tagged n-cells (I,t) in F. Then $\overline{\int} T \leq \Sigma_{k\varepsilon N} \overline{\int} T_k$.

PF. Given q take r as hypothesized and let $S_k(I,t) = T_k(I,t)$ for $k \leq r(t)$, 0 for $k > r(t)$. Then $qT \leq \Sigma_{k\varepsilon N} S_k$. So for every gauge δ on F

(9) $$q\overline{\int} T \leq \Sigma_{k\varepsilon N} \overline{\Sigma}(S_k,\delta).$$

Let ε be given in $\mathbb{R}_+$. Since $0 \leq S_k \leq T_k$ we can choose gauges δ_k on F small enough so that

(10) $$\overline{\Sigma}(S_k,\delta_k) \leq \overline{\int} T_k + \varepsilon/2^k \quad \text{for all } k \text{ in } \mathbb{N}.$$

Let $\delta(t)$ be the minimum of $\delta_k(t)$ for $k = 1,\ldots,r(t)$. By LEMMA A

(11) $$\overline{\Sigma}(S_k,\delta) \leq \overline{\Sigma}(S_k,\delta_k) \quad \text{for all } k \text{ in } \mathbb{N}.$$

By (9),(11),(10) $q\overline{\int} T \leq \Sigma_{k\varepsilon N} \overline{\int} T_k + \varepsilon$. Letting $\varepsilon \to 0+$ and then $q \to 1-$ we get LEMMA B.

THEOREM 7. Let $S \geq 0$ be an m-summant on F. Let $v, v_1, v_2, \ldots \geq 0$ be 1-functions on F such that $v \leq \Sigma_{k\varepsilon N} v_k$. Then

(12) $$\overline{\int} vS \leq \Sigma_{k\varepsilon N} \overline{\int} v_k S.$$

If moreover $\int v_k S$ exists for all k in N and $v = \Sigma_{k\varepsilon N} v_k$ then $\int vS = \Sigma_{k\varepsilon N} \int v_k S$.

PF. We may assume $m=1$ since this case can be applied to each coordinate.

Apply LEMMA B with $T = vS$ and $T_k = v_k S$ to get (12). The final statement follows from (12) and the trivial reversed inequality for the lower integrals.

THEOREM 8. Let T be an m-summant on F such that $T \sim 0$. Then $yT \sim 0$ for every 1-function y on F.

PF. Apply THEOREM 7 with $v = |y|$, $v_k = 1$, and $S = |T|$ to conclude that $yT \sim 0$ from (12).

For E a subset of F let 1_E be the indicator of E, $1_E(t) = 1$ if $t \in E$, 0 if $t \in F \setminus E$. Since indicators are bounded $1_E\sigma = [1_E S]$ is effectively defined for any m-differential $\sigma = [S]$ on F. Call E σ-null if $1_E\sigma = 0$. A condition on the points p of F holds σ-everywhere (or for σ-all p) if it holds at every point p in $F \setminus E$ for some σ-null E. We also use this terminology for any summant S representing σ. By THEOREM 7 a union of countably many σ-null sets is σ-null.

THEOREM 9. Let T be an m-summant and z a 1-function on F. Then $zT \sim 0$ if and only if $z = 0$ T-everywhere.

PF. For k in $\mathbb{N}$ let A_k consist of all points in F at which $|z| > 1/k$. Let A consist of all points where $z \neq 0$. Then $1_A \leq \Sigma_{k\in\mathbb{N}} 1_{A_k}$. Since $1_{A_k} \leq k|z|$, $|1_{A_k}T| \leq k|zT|$. So $zT \sim 0$ implies $1_{A_k}T \sim 0$. Hence (12) in THEOREM 7 with $S = |T|$, $v = 1_A$, and $v_k = 1_{A_k}$ implies $1_A T \sim 0$. Conversely $1_A T \sim 0$ implies $z1_A T \sim 0$ by THEOREM 8. Hence $zT \sim 0$ since $z1_A = z$.

We can now effectively multiply an m-differential $\sigma = [S]$ on F by any 1-function y that is defined σ-everywhere on F. In particular y may be an extended real-valued function on F that is finite σ-everywhere. Define $y\sigma = [uS]$ where u is any 1-function on F such that $u = y$ σ-everywhere. To show effectiveness let $T \sim S$ and v be any 1-function on F with $v = y$ σ-everywhere. Then $v = u$ σ-everywhere and $|uS - vT| \leq |(u-v)S| + |v|\ |S-T| \sim 0$ by THEOREM 8 and THEOREM 9. So $uS \sim vT$. We can now formulate a monotone convergence theorem.

THEOREM 10. Let $\sigma \geq 0$ be an m-differential on F. Let $\langle y_k \rangle$ be a sequence of 1-functions defined σ-everywhere on F such that $0 \leq y_k \leq y_{k+1}$ σ-everywhere, $y_k\sigma$ is integrable, and $\int y_k\sigma \uparrow q$ in $\mathbb{R}^m$. Then $y_k \uparrow y$ σ-everywhere, where y is finite σ-everywhere and $\int y\sigma = q$.

PF. We may assume the σ-everywhere hypothesis holds everywhere on F. Let A consist of all points in F at which $y = \infty$. Let $y_0 = 0$ and $v_k = y_k - y_{k-1}$ for all k in $\mathbb{N}$. Then $y_j = \Sigma_{k=1}^{j} v_k$ so $r1_A \leq \Sigma_{k\in\mathbb{N}} v_k$ for all r in $\mathbb{N}$ since the series diverges on A. By (12) in THEOREM 7 with $v = r1_A$ and $S \geq 0$ representing σ we find that $r\int 1_A S \leq \Sigma_{k\in\mathbb{N}} \int v_k S = q$. Hence $1_A\sigma = 0$ since r can be arbitrarily large. Thus for $B = F\backslash A$, $1_B\sigma = \sigma$. Let $v = y$ on B and 0 on A. Then $v = y$ σ-everywhere, $0 \leq v < \infty$ on F, and $v = \Sigma_{k\in\mathbb{N}} v_k 1_B$. Also $\int v_k S = \int v_k 1_B S$ since $S \sim 1_B S$ implies that $v_k S \sim v_k 1_B S$ by THEOREM 8. So THEOREM 7 gives $\int y\sigma = \int vS = \Sigma_{k\in\mathbb{N}} \int v_k 1_B S = \Sigma_{k\in\mathbb{N}} \int v_k\sigma = q$.

5. ABSOLUTELY INTEGRABLE DIFFERENTIALS. Given an m-differential σ on an n-cell K call a subset E of K σ-measurable if $1_E\sigma$ is absolutely integrable on K. From our foregoing results it easily follows that the σ-measurable subsets of K form a sigma-ring, a sigma-algebra if σ is absolutely integrable on K. For such σ the Borel sets in K are σ-measurable, a result of THEOREM 11.

THEOREM 11. Let σ be an absolutely integrable m-differential on an n-cell K. Then every n-figure A in K is σ-measurable and

$$(13) \qquad \int_K 1_A\sigma \quad = \quad \lim_{B\downarrow A} \int_B \sigma$$

where the limit is taken on the filterbase $\mathfrak{M}$ of all n-figures B in K whose interiors relative to K contain A.

PF. Considering $\sigma+$, $\sigma-$ separately we may in effect assume $\sigma > 0$. Then the limit in (13) exists as an infimum. Given B in $\mathfrak{M}$ take a gauge δ on K fine enough so that for every δ-fine (I,t): (i) I is disjoint from A if t is not in A, (ii) I is contained in B if t is in A. Let $\hat{S}$ be the additive summant representing σ. Given any δ-division $\mathcal{K}$ of K let D be the union of those cells in $\mathcal{K}$ whose tags lie in A. So $\Sigma(1_A\hat{S},\mathcal{K}) = \int_D \sigma$. Also $D \in \mathfrak{M}$ by (i) and $D \subseteq B$ by (ii). Hence $\inf_{B\in\mathfrak{M}} \int_B\sigma <$ $\Sigma(1_A\hat{S},\mathcal{K}) < \int_B \sigma$ which gives (13).

We can now explore the connection between finite Borel measures and positive integrable 1-differentials. Each such differential induces a measure and each measure arises in this way. But distinct differentials may induce the same measure. This occurs because mass distributed on a coordinate hyperplane H by the measure can be apportioned additively by the differential to the two closed half-spaces which share the boundary H. We shall give a simple example of this shortly.

Let $\sigma > 0$ be an integrable 1-differential on K. Let $\mathbb{L}(\sigma)$ be the Riesz space of all 1-functions y on K with $y\sigma$ absolutely integrable. By our foregoing results σ induces a complete Daniell integral $\tilde{\sigma}$ on $\mathbb{L}(\sigma)$ defined by

$$(14) \qquad \tilde{\sigma}(y) = \int_K y\sigma.$$

As is well known this coincides with the Lebesgue-Stieltjes integral of y against the measure $\mathbb{M}$ defined by $\mathbb{M}(E) = \tilde{\sigma}(1_E)$ on the Borel sets E contained in K. The Riesz space $\mathbb{B}$ of all bounded Borel functions y on K is a subspace of $\mathbf{L}(\sigma)$ in which it is dense under the pseudonorm $\mathfrak{n}(y\sigma) = \int_K |y|\sigma$. To see that $\tilde{\sigma} = \tilde{\tau}$ does not imply $\sigma = \tau$ let K be the 1-cell $[-1,1]$, $\sigma = dx$, and $\tau = dv$ where $x = 1_{[0,1]}$ and $v = 1_{(0,1]}$. Then $\tilde{\sigma}(y) = \tilde{\tau}(y) = y(0)$ for every 1-function y on K by (14). But $\int_0^1 \sigma = 0$ and $\int_0^1 \tau = 1$, so $\sigma \neq \tau$. Indeed $\sigma \wedge \tau = 0$, σ attaching a unit left mass at 0, τ a unit right mass. (Of course there _is_ a one-one correspondence between positive integrable 1-differentials on an n-cell K and finite measures on the Stone space of the boolean algebra $\mathcal{F}(K)$ of n-figures in K. Each point t in K yields $2^{m(t)}$ points in the Stone space where $m(t)$ is the number of coordinate hyperplanes through t that cut K.)

Given a Daniell integral $\mathfrak{m}$ on B there exists a positive integrable 1-differential σ on K such that $\tilde{\sigma}(y) = \mathfrak{m}(y)$ for all y in B. Since $\mathfrak{m}$ corresponds to a finite Borel measure $\mathbb{M}$ on K defined by $\mathbb{M}(E) = \mathfrak{m}(1_E)$ we can construct σ from $\mathbb{M}$. To this end consider any n-cell $I = [p,q]$ contained in $K = [a,b]$. $I = I_1 x \ldots x I_n$ for $I_i = [p_i, q_i]$. Let I_i^* be I_i if $p_i = a_i$, $I_i \backslash p_i$ if $p_i > a_i$. Define $I^* = I_1^* x \ldots x I_n^*$. Define the cell summant S on K by $S(I) = \mathbb{M}(I^*)$. Now $K^* = K$ and for $\mathcal{P}$ a partition of I the sets J^* with J in $\mathcal{P}$ disjointly cover I^*. So S is an additive 1-summant on K. Let $\sigma = [S]$. Then $S(I) = \int_I \sigma$ for every n-cell I in K. By (13) of THEOREM 11 and continuity of $\mathbb{M}$, $\int_K 1_I\sigma = \lim_{J \downarrow I} S(J) = \lim_{J\downarrow I} \mathbb{M}(J^*) = \mathbb{M}(I)$ since J^* is a neighborhood of I whenever J is. That is, $\tilde{\sigma}(1_I) = \mathbb{M}(I)$ for every n-cell I in K. So by uniqueness of the Daniell completion we conclude that $\tilde{\sigma}(y) = \mathfrak{m}(y)$ for all y in $\mathbf{L}(\sigma)$.

6. TAG-FINITE, TAG-BOUNDED, AND TAG-NULL DIFFERENTIALS. To simplify notation we denote the singleton $[p,p]$ in $\mathbb{R}^m$ by p. An m-differential σ on F is _tag-finite_ on a subset A of F if $\mathfrak{m}(1_p\sigma) < \infty$ for all p in A. Equivalently there exists an m-function w on F such that

$$(15) \qquad \int_F 1_p|\sigma| < w(p) \text{ for all } p \text{ in } A.$$

σ is _tag-bounded_ on A if (15) holds for some constant m-function w on F. Clearly every absolutely integrable σ is tag-bounded on F. σ is _tag-null_ on A if $1_p\sigma = 0$ for all p in A. If σ is tag-null on A then σ is tag-bounded on A. $\sigma = (\sigma_1,\ldots,\sigma_m)$ is tag-finite, tag-bounded, or tag-null on A if and only if each σ_i has that property. We apply these terms to S if they hold for $\sigma = [S]$. The integral in (15) can be expressed in terms of a simple limit involving S. For convenience set $S(I,t) = 0$ if I does not overlap F. For tagged n-cells (I,t) with $I = [q,r]$ let $N_i(t)$ indicate $t_i = r_i$ and define

$$(16) \qquad N(I,t) = \sum_{i=1}^{n} N_i(t)\, 2^{n-i}.$$

This enumerates the vertices of I assigning them the values $0,\ldots,2^n-1$. Two n-cells I,J with a common vertex t overlap if and only if $N(I,t) = N(J,t)$. In taking limits of summants the convergence $I \to p$ refers to all p-tagged n-cells (I,p) filtered by the neighborhoods of p. The notation $N=k$ adds the restriction that $N(I,p) = k$. Let (J,p) be a tagged n-cell in F. Let $k = N(J,p)$. Since $t = p$ for all sufficiently gauge-fine (I,t) containing p

$$(17) \qquad \bar{\int}_J 1_p|\sigma| = \overline{\lim}_{\substack{I \to p \\ N=k}} |S(I,p)|.$$

So the right side of (17) is independent of the representative S of σ. By

THEOREM 2 (17) gives

$$(18) \qquad \bar{\int}_F 1_p|\sigma| = \Sigma_{k=o}^{2^n-1} \overline{\lim}_{\substack{I \longrightarrow p \\ N=k}} |S(I,p)|$$

recalling that $S = 0$ off F.

<u>THEOREM 12</u>. An m-differential $\sigma = [S]$ on F is tag-null, tag-finite, or tag-bounded on a subset A of F if and only if $\overline{\lim}\ |S(I,p)|$ as $I \longrightarrow p$ is respectively 0, finite, or bounded for all p in A.

PF. By (18) the inequality (15) implies

$$(19) \qquad \overline{\lim}_{I \longrightarrow p} |S(I,p)| < w(p) \quad \text{for all } p \text{ in } A.$$

Conversely (19) implies by (18) that (15) holds for $2^n w$.

Applying THEOREM 12 with $S = \Delta x$ for an m-function on F we see that x bounded implies dx tag-bounded on F. If dx is tag-finite on F and x is bounded on each coordinate hyperplane in F then x is bounded on F. If x is continuous at p then dx is tag-null at p. The converse holds if the restriction of x to the coordinate hyperplanes through p is continuous at p.

7. PRODUCTS WITH DIFFERENTIAL FACTORS. For S a tag-finite 1-summant on F and $\tau = [T]$ a 1-differential on F define

$$(20) \qquad S\tau = [ST], \text{ a 1-differential on } F.$$

To see that (20) is effective let $T' \sim T$. Then $|ST - ST'| = |S|\ |T - T'| \leq w\ |T - T'| \sim 0$ by (19) and THEOREM 8. If τ is not tag-finite then $S' \sim S$ need not imply $S'\tau = S\tau$. But the implication does hold under tag-finiteness. For tag-finite 1-differentials σ, τ on F define the 1-differential

(21) $\sigma\tau = [ST]$ for $\sigma = [S]$ and $\tau = [T]$ on F.

To verify effectiveness choose w so that (15) holds on F for σ and τ. If $S' \sim S$ and $T' \sim T$ then $|ST - S'T'| \leq |S|\,|T - T'| + |T'|\,|S - S'| \leq w|T - T'| + w\,|S - S'| \sim 0$. The products (20), (21) extend by iteration to arbitrarily many factors as long as the appropriate factors are tag-finite. Such products are associative and distributive. We can extend (21) to construct multi-linear products of higher dimensional tag-finite differentials on F homomorphically from corresponding products of representative summants(e.g. exterior products). For differentials defined by products the factors often involve summants that are boosted onto higher dimensional cells from cells of lower dimension by projection of cartesian products. Specifically consider the projection of an n-cell JxL onto the q-cell J that takes $t = (t_1,\ldots,t_q,\ldots,t_n)$ to $t' = (t_1,\ldots,t_q)$. Each tagged n-cell (I,t) in JxL projects onto a tagged q-cell (I',t') in J. Each m-summant R on J thereby induces an m-summant S on JxL defined by $S(I,t) = R(I',t')$. One must take care not to confuse the m-differential $\sigma = [S]$ on JxL with the m-differential $\rho = [R]$ on J. (Indeed there are cases where $\rho = 0$ but $m(\sigma) = \infty$.) Typically let x_i be a bounded 1-function on a 1-cell K_i for $i = 1,\ldots,n$. Define a 1-function x on $K = K_1 x\ldots x K_n$ by $x(t) = x_1(t_1)\ldots x_n(t_n)$ for $t = (t_1,\ldots,t_n)$. So $\Delta x(I) = \Delta x_1(I_1)\ldots\Delta x_n(I_n)$ for any n-cell $I = I_1 x\ldots x I_n$ in K where I_i is the projection of I into K_i. The traditional notation expresses this as $dx = dx_1\ldots dx_n$. But as a product of differentials $dx = \sigma_1\ldots\sigma_n$ with the 1-differential $\sigma_i = [S_i]$ on K defined by the boosted cell summant $S_i(I) = \Delta x_i(I_i)$ on K. Here σ_i is determined by dx_i but must not be identified with it. We should use some such notation as $\sigma_i = \overline{dx}_i$.

8. NEGLIGIBLE SETS AND DAMPABLE DIFFERENTIALS. A set Q of tagged n-cells

in F is <u>σ-negligible</u> for an m-differential σ on F if $Q\sigma = 0$ for Q the 1-summant on F that indicates $\sim Q$. A subset A of F is σ-null if and only if the set of all tagged n-cells in F with tags in A is σ-negligible. A <u>damper</u> is a function u on F into $\mathbb{R}_+$. A 1-differential σ on F is <u>dampable</u> if $u\sigma$ is absolutely integrable for some damper u. An m-differential is <u>dampable</u> if each component is dampable. An m-differential σ is <u>summable</u> if $\mathbf{n}(\sigma) < \infty$. σ is <u>damper-summable</u> if $u\sigma$ is summable for some damper u. Equivalently σ is damper-summable if F is the union of countably many sets E with $1_E\sigma$ summable. For $\sigma = dx$ on a 1-cell K this resembles the condition that x be VBG* [18]. But VBG*-functions may be unbounded on K whereas x must be bounded if dx is damper-summable on K. Indeed every damper-summable differential is tag-finite. There even exist bounded VBG*-functions whose differentials are not damper-summable (e.g. the indicator of the rationals in K). If a 1-function y on a 1-cell K has finite derivates dy-everywhere then dy is damper-summable. Since dampers need not be measurable there are some open questions: Is dx dampable if it is damper-summable, or even if $|dx|$ is dampable?

Damper-summable differentials σ are a boon to analysis because they are archimedean: If $|\rho| < \varepsilon|\sigma|$ for all ε in $\mathbb{R}_+$ then $\rho = 0$. Our next result exploits this property. It will be used to get differentials from derivatives.

<u>THEOREM 13.</u> Let $\rho = [R]$ and $\pi = [P]$ be 1-differentials on F such that (i) π is damper-summable, (ii) given ε in $\mathbb{R}_+$ the set of all tagged n-cells (I,t) in F for which

(22) $$|R(I,t)| > \varepsilon|P(I,t)|$$

is ρ-negligible. Then $\rho = 0$. Condition (ii) is implied by:

(iii) for ρ-all t in F

$$R(I,t) = o(P(I,t)) \quad \text{as} \quad I \longrightarrow t \quad \text{in } F. \tag{23}$$

PF. Given ε in $\mathbb{R}_+$ let Q indicate (22). Then $(1-Q)R \sim R$ since $QR \sim 0$. Also $|(1-Q)R| < \varepsilon|P|$. So $|R| \leq \varepsilon|P|$. That is, $|\rho| < \varepsilon|\pi|$. By (i) this implies $\rho = 0$ since ε is arbitrary. Given (iii) let A consist of all t for which (23) holds. Given ε in $\mathbb{R}_+$ choose a gauge δ on F such that (23) gives $|R| < \varepsilon|P|$ on all δ-fine tagged n-cells in F with tags in A. $F \setminus A$ is ρ-null by (iii). So the set of all tagged n-cells with tags in $F \setminus A$ is ρ-negligible. The set of all tagged n-cells which are not δ-fine is obviously ρ-negligible. Since the union of two ρ-negligible sets is ρ-negligible, (ii) follows from (iii).

Applied to derivatives (iii) is useful only for $\dim F = 1$.

9. THE CHAIN RULE ON 1-CELLS. We shall present two formulations for the chain rule on 1-cells. Each gives the classical formulas of differential calculus. We denote the inner product in $\mathbb{R}^m$ by a dot between the two factors. Following [3] a 1-function f on a neighborhood of x in $\mathbb{R}^m$ is <u>differentiable</u> at x if there exists for x an m-function g on a neighborhood V of 0 in $\mathbb{R}^m$ such that g is continuous at 0 and

$$f(x+h) - f(x) = g(h) \cdot h \quad \text{for all } h \text{ in } V. \tag{24}$$

Then $g(0)$ gives the gradient of f at x, $g(0) = \operatorname{grad} f\,(x) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)(x)$. We shall prove THEOREM 14,15 together under the hypothesis (H): Let x be a continuous m-function on a 1-cell K with dx damper-summable, f a 1-function on a neighborhood of the curve $x(K)$ in $\mathbb{R}^m$, and $y(t) = f(x(t))$ for all t in K.

THEOREM 14. Given (H) with f differentiable at $x(t)$ except for countably many t in K, and y continuous on K, let $z(t) = \text{grad } f(x(t))$ wherever f is differentiable. Then $dy = z \cdot dx$.

THEOREM 15. Given (H) with f differentiable at $x(t)$ for dy-all t in K let $z(t) = \text{grad } f(x(t))$ if f is differentiable at $x(t)$, and $z(t) = 0$ otherwise. Then y is continuous and $dy = z \cdot dx$.

PF. Let D be the set of all t in K with f differentiable at $x(t)$. Given t in D set $x = x(t)$ in (24). Given a 1-cell I in K with endpoint t let $t+q$ be the other endpoint. Set $h = x(t+q) - x(t) = (\text{sgn } q)\Delta x(I)$. Then $f(x+h) - f(x) = y(t+q) - y(t) = (\text{sgn } q)\Delta y(I)$. So (24) multiplied through by $\text{sgn } q$ gives $\Delta y(I) = g(h)\cdot\Delta x(I)$. Hence, since $g(0) = z(t)$, $\Delta y(I) - z(t)\cdot\Delta x(I) = (g(h)-g(0))\cdot\Delta x(I)$. Since x is continuous $h \to 0$ as $I \to t$. Thus, since g is continuous at 0, $\Delta y(I) - z(t)\cdot\Delta x(I) = o(||\Delta x(I)||)$ as $I \to t$ in K for all t in D. That is, (23) holds for $R = \Delta y - z\cdot\Delta x$ and $P = ||\Delta x||$ at all t in D. So we need only show that the complement C of D in K is ρ-null for $\rho = dy - z\cdot dx$ to get THEOREM 14,15 from THEOREM 13.

In THEOREM 14 C is countable, hence dx-null and dy-null since x and y are continuous. So C is also $z\cdot dy$-null. Thus C is ρ-null.

In THEOREM 15 C is dy-null by hypothesis. C is $z\cdot dx$-null since $1_C z = 0$ by the definition of z. So C is ρ-null. Since $dy = z\cdot dx$ with x continuous, y must be continuous because $1_p dy = z\cdot 1_p dx = 0$ for all p in K.

For x a continuous 1-function on a 1-cell K with dx damper-summable and $y = f(x)$ with f a differentiable 1-function we get the classical fundamental theorem of calculus, $dy = f'(x)dx$.

Let u,v be continuous 1-functions on a 1-cell K with du,dv damper-summable. Then the 2-function $x = (u,v)$ has the damper-summable differential $dx = (du,dv)$. For $f(x) = uv$ we have $z = \text{grad } f = (v,u)$. So the chain rule $dy = z \cdot dx$ gives the product rule $d(uv) = vdu + udv$. If v has no zeros on K we can apply the chain rule with $f(x) = u/v$ to get $d(u/v) = (vdu - udv)/v^2$. Note that $dy = z \cdot dx$ concisely formulates the statement that $z \cdot dx$ is integrable on K and $\Delta y(I) = \int_I z \cdot dx$ for every 1-cell I in K.

10. DERIVATIVES ON 1-CELLS. Let x be a 1-function and y an m-function on a 1-cell K. Define $\frac{dy}{dx}(t) = \lim \frac{\Delta y(I)}{\Delta x(I)}$ as $I \to t$ in K wherever the limit exists. Equivalently $\frac{dy}{dx}(t) = \lim_{h \to 0} \frac{y(t+h)-y(t)}{x(t+h)-x(t)}$ under the restriction that $h \neq 0$ and $t+h \in K$. Existence of this limit requires that $\Delta x(I) \neq 0$ for all sufficiently small 1-cells I in K with endpoint t. It does not require that $\Delta x(I) \to 0$ as $I \to t$. Indeed $(dy/dx)(t)$ may exist at some t where x is discontinuous. Similar definitions hold for $dy/|dx|$, $|dy|/dx$, and $|dy|/|dx|$ as the respective limits of $\Delta y/|\Delta x|$, $|\Delta y|/\Delta x$, and $|\Delta y|/|\Delta x|$. From our point of view these derivatives are not quotients of differentials despite the traditional Leibnitz notation. But with mild restrictions on x there is a connection between $dy/dx = z$ and $dy = z\,dx$. One such restriction involves "balance". An m-differential σ on an n-cell K is <u>balanced at p</u> in K if $1_p\sigma = 0$ whenever $\int_I 1_p\sigma = 0$ for some n-cell I in K with vertex p. σ is <u>balanced</u> if it is balanced at p for all p in K. THEOREM 16 will be used to get derivatives from differentials on 1-cells. It is restricted to 1-cells because the corona property that gives the Vitali Covering Theorem for Borel measures in one dimension fails in higher dimensions [4].

<u>THEOREM 16</u>. Let $\sigma, \rho > 0$ be 1-differentials on a 1-cell K such that σ is integrable and balanced, ρ is tag-finite, and $\rho\sigma = 0$. Then ρ is tag-null

σ-everywhere.

PF. Let $\sigma = [S]$, $\rho = [R]$ with $S > 0$, $R > 0$. Given n in $\mathbb{N}$ let C be the set of all p in K at which $\overline{\lim}\, R(I,p) > 1/n$ as $I \to p$ in K. We need only show that C is σ-null to conclude that $R(I,p) \to 0$ σ-everywhere as $I \to p$. Since $\rho\sigma = 0$, $1_p \rho\sigma = 0$ so $1_p RS \sim 0$ for all p in K. Hence $(SR)(I,p) \to 0$ as $I \to p$ in K. So for all p in C, $\underline{\lim}\, S(I,p) = 0$ as $I \to p$ in K. Since $\sigma > 0$ is balanced and integrable this implies σ is tag-null on C. Given ε in $\mathbb{R}_+$ the integrability of $\sigma > 0$ implies the existence of a finite (possibly empty) subset D of K such that $1_p\sigma > 0$ for all p in D and

(25) $$\Sigma_{p \varepsilon K \setminus D} \int_K 1_p \sigma < \varepsilon.$$

Since σ is tag-null on C but nowhere on D, $C \cap D = \emptyset$. Take a gauge δ on K small enough so that for every δ-division $\mathcal{K}$ of K

(26) $$|\Sigma(S,\mathcal{K}) - \int_K \sigma| < \varepsilon$$

and

(27) $$\Sigma(RS, \mathcal{K}) \leq \varepsilon.$$

Let $\mathcal{C}$ be the set of all 1-cells I in K disjoint from D with an end-point p such that

(28) $$p \in C,\ (I,p) \text{ is } \delta\text{-fine, and } R(I,p) > 1/n.$$

The following version of Vitali Covering Theorem [4] is now applicable: Let C be a subset of a 1-cell K. Let $\mathcal{C}$ be a set of 1-cells in K such that given p in C and a neighborhood G of p, p belongs to some member of $\mathcal{C}$ contained in G. Then given an integrable 1-differential $\sigma > 0$ on K there exists a (countable) subset $\mathcal{D}$ of $\mathcal{C}$ whose members are disjoint and cover σ-all of C. We thus get a σ-null set E with $1_C < 1_E + \Sigma_{I\in\mathcal{D}}\, 1_I$. Therefore $\bar{\int}_K 1_C\sigma < \int_K(\Sigma_{I\in\mathcal{D}}\, 1_I\,)\sigma = \Sigma_{I\in\mathcal{D}} \int_K 1_I\sigma < \Sigma_{I\in\mathcal{D}} \int_I \sigma + \Sigma_{p\in K\setminus D'} \int_K 1_p\sigma < \Sigma_{I\in\mathcal{D}} \int_I \sigma +$ ε by THEOREM 7, (13) in THEOREM 11, and (25). That is,

(29) $$\bar{\int}_K 1_C\sigma < \Sigma_{I\in\mathcal{D}} \int_I \sigma + \varepsilon\ .$$

To each member I of $\mathcal{D}$ assign a tag p satisfying (28) to get a set $\mathcal{E}$ of tagged 1-cells. By (26) and THEOREM 3 applied to the corresponding partial sums over $\mathcal{D}$ and $\mathcal{E}$

(30) $$\Sigma_{I\in\mathcal{D}} \int_I \sigma < \Sigma(S,\mathcal{E}) + \varepsilon\ .$$

By (28) $nR > 1$ on $\mathcal{E}$. So $\Sigma(S,\mathcal{E}) < n\Sigma(RS,\mathcal{E}) < n\varepsilon$ by (27). That is,

(31) $$\Sigma(S,\mathcal{E}) < n\varepsilon.$$

From (29), (30), (31) we conclude $\bar{\int}_K 1_C\sigma < (n+2)\varepsilon$ for all ε in $\mathbb{R}_+$. Hence $1_C\sigma = 0$ since ε is arbitrary.

<u>THEOREM 17</u>. Let x be a 1-function on a 1-cell K with $|dx|$ dampable and balanced. Let y,z be m-functions on K. Let (c) denote the condition that every dx-null set in K is dy-null. Then (a_i) is equivalent to (b_i) for $i = 1,2,3,4$:

(a_1) $dy = z\,dx$, (b_1) $dy/dx = z$ dx-everywhere and (c) holds,

(a_2) $dy = z|dx|$, (b_2) $dy/|dx| = z$ dx-everywhere and (c) holds,

(a_3) $|dy| = z\,dx$, (b_3) $|dy|/dx = z$ dx-everywhere and (c) holds,

(a_4) $|dy| = z|dx|$, (b_4) $|dy|/|dx| = z$ dx-everywhere and (c) holds.

PF. Given (a_1) choose a damper u with $u|dx|$ integrable. Let $S = u|\Delta x|$

and $\sigma = [S] = u|dx|$. Given ε in $\mathbb{R}_+$ let R be the summant that indicates $||\Delta y - z\Delta x||_1 > \varepsilon S$. Then $0 < \varepsilon RS < ||\Delta y - z\Delta x||_1$. For $\rho = [R]$ this is just $0 < \varepsilon\rho\sigma < ||dy - z\, dx||_1 = 0$ by (a_1). So $\rho\sigma = 0$. By THEOREM 16 $R(I,p) = 0$ ultimately as $I \to p$ in K for σ-all p. dx has the same null sets as σ. So by the definition of R, $||\Delta y(I) - z(p)\Delta x(I)||_1 < \varepsilon u(p)|\Delta x(I)|$ ultimately as $I \to p$ for dx-all p. This inequality is just $||(\Delta y/\Delta x)(I) - z(p)||_1 < \varepsilon u(p)$. So $dy/dx = z$ dx-everywhere. For A dx-null $1_A\, dy = z1_A\, dx = 0$ by (a_1) giving (c). So (a_1) implies (b_1). Conversely let (b_1)hold. Apply THEOREM 13 with $R = ||\Delta y - z\Delta x||_1$ and $P = |\Delta x|$. Then $\rho = ||dy - zdx||_1$ and $\pi = |dx|$. The condition $(dy/dx)(t) = z(t)$ implies (23). So (23) holds dx-everywhere by (b_1), hence zdx-everywhere. By (c) (23) holds dy-everywhere. So (23) holds (dy-zdx)-everywhere, hence ρ-everywhere. That is, (iii) holds in THEOREM 13 which implies $\rho = 0$. So (b_1) implies (a_1). Similar proofs hold for $i = 2,3,4$ if we replace Δx by $|\Delta x|$ (dx by $|dx|$) and/or Δy by $|\Delta y|$ (dy by $|dy|$).

Since $|dx|$ is dampable the condition that it be balanced means just that at every interior point p of K x is left continuous at p if and only if x is right continuous at p. Our next result is similar to THEOREM 15 but requires neither continuity of x nor explicit funtional dependence of y on x.

<u>THEOREM 18.</u> Let x be a 1-function on a 1-cell K with dx damper-summable. Let y be an m-function on K such that dy/dx exists dy-everywhere. Define $z(t) = (dy/dx)(t)$ wherever the derivative exists, and $z(t) = 0$ elsewhere. Then $dy = z\, dx$.

PF. Apply THEOREM 13 with $R = ||\Delta y - z\Delta x||_1$ and $P = \Delta x$. So $\rho =$

$||dy - zdx||_1$ and $\pi = dx$. (23) holds at all t where $(dy/dx)(t)$ exists, hence dy-everywhere. Since $z = 0$ where the derivative does not exist, dy/dx exists zdx-everywhere. So (23) holds zdx-everywhere, hence ρ-everywhere. Thus THEOREM 13 gives $dy = zdx$.

As in THEOREM 17 there are variants of THEOREM 18 with dx replaced by $|dx|$ and/or dy by $|dy|$.

11. RADON-NIKODYM DIFFERENTIAL COEFFICIENTS. Let σ be an integrable m-differential on an n-cell K. For each n-figure F in K define the projection σ_F to be the integrable m-differential $[S_F]$ where S_F is the additive m-summant defined by $S_F(I) = \int_{I \cap F} \sigma$ for each n-cell I in K. The integral over degenerate figures is zero. Application of Bochner's Step Function Density Theorem [2] to additive summants yields the differential formulation THEOREM 19. (See THEOREM 9 in [13] with $p=1$ and THEOREM 1 and section 9 in [14] which give THEOREM 19 for $\sigma > 0$. This special case easily extends to the case of absolutely integrable σ.)

<u>THEOREM 19</u>. Let σ be an absolutely integrable 1-differential on an n-cell K. Let $\mathbb{V}$ consist of all integrable 1-differentials τ on K such that on n-figures F in K

(32) $$\int_F \tau \to 0 \quad \text{as} \quad \int_F |\sigma| \to 0.$$

Then each τ in $\mathbb{V}$ is absolutely integrable, $\mathbb{V}$ is an (L)-space (a Banach lattice with norm additive on the positive cone [10]) under the norm $\mathbf{n}(\tau) = \int_K |\tau|$. The linear subspace of $\mathbb{V}$ generated by the projections σ_J of σ for all n-cells J in K is dense in $\mathbb{V}$ under the norm topology.

We use this result to prove a Radon-Nikodym theorem for 1-differentials on n-cells.

THEOREM 20. Let τ,σ be dampable 1-differentials on an n-cell K such that (i) every coordinate hyperplane that cuts K is σ-null, and (ii) every σ-null subset of K is τ-null. Then $\tau = z\sigma$ for some 1-function z on K.

PF. We first treat the case where τ,σ are absolutely integrable. So weak absolute continuity (ii) implies strong absolute continuity on Borel sets E in K,

$$(33) \qquad \int_K 1_E|\tau| \to 0 \quad \text{as} \quad \int_K 1_E|\sigma| \to 0.$$

Given an n-figure F in K let E be the interior of F relative to K. By (i), $1_E = 1_F$ σ-everywhere. So $\int_F |\sigma| = \int_F 1_F|\sigma| = \int_F 1_E|\sigma| = \int_K 1_E|\sigma|$. A similar result holds for τ by (i),(ii). So (33) gives (32). For every n-cell I in K, $\int_I 1_F\sigma = \int_I 1_E\sigma = \int_{I\cap F} 1_E\sigma = \int_{I\cap F} 1_F\sigma = \int_{I\cap F} \sigma = S_F(I) = \int_I \sigma_F$. So $1_F\sigma = \sigma_F$ by THEOREM 5. In particular $\sigma_J = 1_J\sigma$. So σ_J belongs to the closed subspace of $\mathbb{V}$ consisting of all absolutely integrable $y\sigma$ with y a 1-function on K. Thus THEOREM 19 gives THEOREM 20 for σ,τ absolutely integrable. For the more general case let u,v be dampers with $u\sigma,v\tau$ absolutely integrable. Apply the previous case to $u\sigma,v\tau$. Since a damper is nowhere zero σ and $u\sigma$ have the same null sets. Similarly so do τ and $v\tau$. Thus the hypothesis (i), (ii) for σ,τ holds as well for $u\sigma,v\tau$ yielding the conclusion that $v\tau = zu\sigma$ for some z. So $\tau = y\sigma$ for $y = zu/v$.

THEOREM 21. Let x be a continuous m-function on an n-cell K with $|dx|$ dampable. Then every coordinate hyperplane H is dx-null.

PF. We may assume $m = 1$ since this case can be applied to each component of x. Let u be a damper with $u|dx|$ integrable. Define v on $K = [a,b]$ by

$v(t) = \int_a^t u|dx|$. (See (8) and THEOREM 6.) Then $dv = u|dx|$ so dv and dx have the same null sets. Let H consist of all t in K with $t_j = c$. We contend H is dv-null. Since x is continuous so is v. Given ε in $\mathbb{R}_+$ take h in $\mathbb{R}_+$ so that the uniform continuity of v on K gives

(34) $|v(t) - v(s)| < \varepsilon/2^n$ for all s,t in K such that $||t - s||_1 < 2h$.

Let the n-cell J consist of all t in K with $c-h < t_j < c+h$. Then $\Delta v(J) < 2^{n-1}(\varepsilon/2^n) < \varepsilon$ by (6),(34). Since H is interior to J relative to K, $\int_K 1_H dv < \int_J dv = \Delta v(J) < \varepsilon$. Thus since ε is arbitrary we conclude that $1_H dv = 0$.

<u>THEOREM 22.</u> Let x,y be continuous 1-functions on an n-cell K such that dx,dy are dampable and every dx-null set is dy-null. Then $dy = z\,dx$ for some 1-function z on K.

PF. Apply THEOREM 21 and THEOREM 20.

The Radon-Nikodym Theorem is related to the Hahn Decomposition Theorem. The connection here comes from THEOREM 23.

<u>THEOREM 23.</u> Let $\rho,\pi > 0$ be m-differentials on an n-cell K. Let $x,y > 0$ be 1-functions on K. Then $(x\rho) \wedge (y\pi) < (x \vee y)(\rho \wedge \pi)$.

PF. Let $\phi = (x\rho) \wedge (y\pi)$. Then $0 < \phi < x\rho$. So $x > 0$ ϕ-everywhere and $(1/x)\phi < \rho$. Similarly $(1/y)\phi < \pi$. So $(\frac{1}{x \vee y})\phi = (\frac{1}{x} \wedge \frac{1}{y})\phi = (\frac{1}{x}\phi) \wedge (\frac{1}{y}\phi) < \rho \wedge \pi$. Multiply through by $x \vee y$ to get THEOREM 23.

THEOREM 24. Let σ be a dampable 1-differential on an n-cell K in which every coordinate hyperplane is σ-null. Then there are complementary subsets A,B of K such that $1_A\sigma^- = 1_B\sigma^+ = 0$.

PF. Apply THEOREM 20 with $\tau = \sigma^+$ to get $\sigma^+ = z\sigma$ for some 1-function z on K. This implies $(z^2-z)\sigma^+ = z^2\sigma^-$. Apply THEOREM 23 with $\rho=\sigma^+$, $\pi=\sigma^-$, $x=|z^2-z|$, $y=z^2$ to conclude from $\sigma^+\wedge\sigma^- = 0$ that $(z^2-z)\sigma^+ = z^2\sigma^- = 0$. Thus $z = 0,1$ σ^+-everywhere and $z=0$ σ^--everywhere. Let $A = z^{-1}(1)$ and $B = K\setminus A$. Apply THEOREM 9 to get THEOREM 24.

THEOREM 25. Let x be a continuous 1-function on an n-cell K with dx dampable. Then there are complementary subsets A,B of K such that $1_A(dx)^- = 1_B(dx)^+ = 0$.

PF. Apply THEOREM 21 and THEOREM 24.

12. PRESERVING DAMPABILITY. Given an absolutely integrable m-differential π on an n-cell K call an m-function z on K π-measurable if $z^{-1}(B)$ is a π-measurable subset of K for every Borel set B in $\mathbf{R}^m$. Borel measurable implies π-measurable.

THEOREM 26. Let σ be a damper-summable m-differential on an n-cell K. Let z be a 1-function on K such that $z\sigma$ is integrable. Then $z\sigma$ is damper-summable. If moreover σ is dampable by a damper u for which z is $u\sigma$-measurable then $z\sigma$ is dampable.

PF. Given a damper u with $\mathfrak{n}(u\sigma) < \infty$ define the damper v on K by $v =$

$u/|z|$ if $z \neq 0$, $v = 1$ if $z = 0$. Then $v|z\sigma| \leq u|\sigma|$. So $m(vz\sigma) \leq m(u\sigma) < \infty$. Thus $z\sigma$ is damper-summable with damper v. Let $u\sigma$ be absolutely integrable and z be $u\sigma$-measurable. Then $(vz\sigma)^+ = (\text{sgn } z)^+u\sigma^+ + (\text{sgn } z)^-u\sigma^-$ and $(vz\sigma)^- = (\text{sgn } z)^-u\sigma^+ + (\text{sgn } z)^+u\sigma^-$ are integrable since each of the four right-hand terms is integrable. So $vz\sigma$ is absolutely integrable.

<u>THEOREM 27</u>. Let x,y,z be 1-functions on a 1-cell K such that $dy = z\,dx$, $|dx|$ is dampable with damper u, and $|dx|$ is balanced. Then z is $u|dx|$-measurable. So $|dy|$ is dampable and balanced. If moreover dx is dampable then so is dy.

PF. Since $dy = z\,dx$ we have $1_p|dy| = |z(p)|1_p|dx|$. So $|dy|$ is balanced because $|dx|$ is balanced. Since $|dx|$ is dampable x can have only countably many points of discontinuity. This conclusion holds also for y since $1_p dy = 0$ wherever $1_p dx = 0$. Let $I_n = [t, t + 1/n] \cap K$ for t in K, n in $\mathbb{N}$. Define z_n on $K = [a,b]$ by $z_n(t) = (\Delta y/\Delta x)(I_n)$ if $\Delta x(I_n) \neq 0$, $z_n(t) = z(b)$ if $\Delta x(I_n) = 0$. The 1-functions $\Delta x(I_n)$ and $\Delta y(I_n)$ of t have only countably many discontinuities. So z_n is Borel measurable, hence $u|dx|$-measurable. By THEOREM 17 there exist complementary subsets A,B of K such that $z_n \to z$ on A, and B is dx-null. Since dx and $u|dx|$ have the same null sets, B and A are $u|dx|$-measurable. So $1_A z$ and $1_B z$ are $u|dx|$-measurable. Hence so is z. So $|dy|$ is dampable by THEOREM 26. Given that $u\,dx$ is absolutely integrable then dy is dampable by THEOREM 26.

<u>THEOREM 28</u>. Let y be a 1-function on a 1-cell K such that dy/dx exists and is finite dy-everywhere for some continuous 1-function x on K whose differential dx is dampable. Then dy is dampable.

PF. Apply THEOREM 18 and THEOREM 27.

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