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BANACH ALGEBRAS OF FUNCTIONS HAVING GENERALIZED BOUNDED VARIATION

Throughout f will be a real valued function defined on a closed interval [a, b]. Extensive use will be made of partitions or subdivisions $x_0, x_1, \ldots, x_n$ of [a, b] for which $a \leq x_0 < x_1 < \ldots < x_n \leq b$. Such subdivisions will be referred to as π-subdivisions.

In 1881, Camille Jordan introduced his well known concept of bounded variation. The total variation of f on [a, b] is defined as

$$V_1(f; a, b) = \sup_{\pi} \sum_{i=1}^{n} \left|f(x_i) - f(x_{i-1})\right|.$$

If $V_1(f; a, b) < \infty$, f is said to be of bounded variation on [a, b], written $f \in BV[a, b]$ or $f \in BV_1[a, b]$.

Many extensions and generalizations of Jordan's variation have been given subsequently. The particular generalization to be discussed here arises from the following result:

In 1911, F. Riesz [2] showed that a real valued function $f: [a, b] \to R$ is an integral of a function of bounded variation if and only if

$$\sum_{i=0}^{n-2} \left| \frac{f(x_{i+2}) - f(x_{i+1})}{x_{i+2} - x_{i+1}} - \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \right| \tag{1}$$

is bounded for all possible π-subdivisions of [a, b]. Functions having property (1) are said to have bounded slope variation. We rename this variation bounded second variation, and extended the notion to bounded k^{th} variation. In order to define bounded k^{th} variation the definition of k^{th} divided difference is required.

k^{th} divided difference The k^{th} divided difference is defined to be

$$Q_k(f; x_0, \ldots, x_k) = \sum_{i=0}^{k} \frac{f(x_i)}{\prod\limits_{\substack{j=0 \\ j \neq i}}^{k} (x_i - x_j)},$$

where $x_0, x_1, \ldots, x_k$ are arbitrary points in [a, b].

We will subsequently have need to refer to k-convex functions in order to characterize functions of bounded k^{th} variation.

k-convex functions A function f is k-convex on [a, b] if and only if

$$Q_k(f; x_0, x_1, \ldots, x_k) \geq 0$$

for all choices of $x_0, x_1, \ldots, x_k$ in [a, b].

There are numerous properties of k^{th} divided differences, many of which are well known. We will mention some here that have been particularly useful in the subsequent work. Further properties can be found in [1].

Properties of k^{th} divided differences

(a) $Q_k(f; x_0, \ldots, x_k) = 0$ for all choices of $x_0, x_1, \ldots, x_k$ iff f is a polynomial of degree (k - 1) at most.

(b) If f is a polynomial of degree k, then $Q_k(f; x_0, \ldots, x_k)$ is equal to the leading coefficient for all choices of $x_0, x_1, \ldots, x_k$.

(c) $Q_k(f; x_0, x_1, \ldots, x_k)$ is independent of the order in which the points $x_0, x_1, \ldots, x_k$ appear.

(d) If $x_0, x_1, \ldots, x_k$ are any k + 1 distinct points of [a, b], then

$$(x_0 - x_k)Q_k(f; x_0, x_1, \ldots, x_k)$$

$$= Q_{k-1}(f; x_0, \ldots, x_{k-1}) - Q_{k-1}(f; x_1, \ldots, x_k).$$

(e) Let $x_0, x_1, \ldots, x_k$ be any k + 1 points of [a, b] such that $x_0 < x_1 < \ldots < x_k$. Suppose that r extra points of subdivision are added to existing subintervals. Then relabelling the points of subdivision as $y_0, y_1, \ldots, y_{k+r}$, where $x_0 = y_0 < y_1 < \ldots < y_{k+r} = x_k$, there exist positive coefficients $\alpha_0, \alpha_1, \ldots, \alpha_r$, independent of f, such that

$$Q_k(f; x_0, \ldots, x_k) = \sum_{i=0}^{r} \alpha_i Q_k(f; y_i, \ldots, y_{i+k}), \text{ and } \sum_{i=0}^{r} \alpha_i = 1.$$

<u>Bounded k^{th} variation</u> The total k^{th} variation of f on [a, b] is defined by

$$V_k(f) \equiv V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} (x_{i+k} - x_i)|Q_k(f; x_i, \ldots, x_{i+k})|.$$

If $V_k(f; a, b) < \infty$, we say that f has bounded k^{th} variation on [a, b] and write $f \in BV_k[a, b]$. In view of Property (d) of k^{th} divided differences, we can write

$$V_k(f; a, b) = \sup_{\pi} \sum_{i=0}^{n-k} |Q_{k-1}(f; x_i, \ldots, x_{i+k-1}) - Q_{k-1}(f; x_{i+1}, \ldots, x_{i+k})|.$$

This form of the total k^{th} variation is more useful, and indeed is the form used for most calculations.

The class of functions $BV_k[a, b]$ has the following properties. The sequence $\{BV_k[a, b]\}$ is contracting, that is $BV_{k+1}[a, b] \subseteq BV_k[a, b]$, and

$$\bigcap_{k=1}^{\infty} BV_k[a, b] = C^{\infty}[a, b],$$

the class of infinitely differentiable functions on [a, b]. Proofs of these results and further properties can be found in [3].

The following characterization of $BV_k[a, b]$ has been obtained:

$BV_k[a, b] = \{f\colon\ f = f_1 - f_2$, where f_1, f_2 are 1-, 2-, ... , k-convex functions having right and left $(k - 1)^{th}$ derivatives at a, b respectively$\}$.

It is the structure of $BV_k[a, b]$ that we are particularly concerned with here. First of all, it is a Banach space under the norm $\|\cdot\|_k$, where

$$\|f\|_k = \sum_{s=0}^{k-1} \left|f_+^{(s)}(a)\right| + V_k(f; a, b). \tag{2}$$

Clearly, $BV_k[a, b]$ is a vector space, for if f and g belong to $BV_k[a, b]$, so does f + g, and

$$V_k(f + g) \leq V_k(f) + V_k(g).$$

Second, $\|\cdot\|_k$ has the properties of a norm, and finally $BV_k[a, b]$ is complete under (2). The completeness is technical, the details of which are given in [4].

We now turn to the Banach algebra aspect of $BV_k[a, b]$, and towards this end we introduce a subspace of $BV_k[a, b]$, namely

$$BV_k^*[a, b] = \{f \in BV_k[a, b]: \ f(a) = f'_+(a) = \ldots = f_+^{(k-1)}(a) = 0\}.$$

$BV_k^*[a, b]$ is clearly a Banach space under the norm

$$\|\cdot\|_k^*, \text{ where } \|f\|_k^* = \alpha_k V_k(f), \text{ and where } \alpha_k = 2^{k-1}(b - a)^{k-1}(k - 1)!$$

The significance of the constant α_k will become apparent later.

The immediate problem is to show that if f, g ε $BV_k^*[a, b]$, then so does fg (pointwise multiplication), and to obtain a suitable inequality relating $V_k(fg)$, $V_k(f)$ and $V_k(g)$. To do this it is convenient, in order to reduce the complexity of calculations, to consider π_h subdivisions of [a, b] which have the form

$$a = x_0 < x_1 < \ldots < x_n \leq b, \text{ where } x_i - x_{i-1} = h, \; i = 1, 2, \ldots, n$$

and $0 \leq b - x_n \leq h$.

Along with π_h subdivisions we consider the well known difference operator Δ_h^k defined by

$$\Delta_h^1 f(x) = f(x + h) - f(x),$$

and

$$\Delta_h^k f(x) = \Delta_h^1 [\Delta_h^{k-1} f(x)].$$

There is obviously a close relationship between $\Delta_h^k f(x)$ and k^{th} divided differences. Indeed, for (k + 1) equally spaced points

$$Q_k(f; x_0, x_1, \ldots, x_k) = \frac{1}{k!} \frac{\Delta_k^h f(x)}{h^k},$$

or

$$\frac{\Delta_h^k (fx_i)}{h^{k-1}} = (k - 1)!(x_{i+k} - x_i)Q_k(f; \; x_i, \ldots, x_{i+k}).$$

Associated with the difference operator is another form of k^{th} variation.

Bounded k^{th} variation (Restricted form)

If $f \in C[a, b]$, define total k^{th} variation on $[a, b]$ by

$$\bar{V}_k(f) \equiv \bar{V}_k(f; a, b) = \sup_{\pi_h} \left|\frac{\Delta_k^h f(x)}{h^{k-1}}\right|.$$

If $\bar{V}_k(f) < \infty$, we say that f has bounded k^{th} variation (restricted form), and write $f \in B\bar{V}_k[a, b]$. As might be expected, there is also a close relationship between $B\bar{V}_k[a, b]$ and $BV_k[a, b]$. Some results relevant to this paper are now given.

Theorem 1 (i) $C[a, b] \cap BV_k[a, b] = B\bar{V}_k[a, b], \ k \geq 1.$

(ii) $\bar{V}_k(f; a, b) = (k - 1)!V_k(f; a, b), \ k \geq 1.$

The restriction to continuous functions is not nearly so severe as might have been expected because if $k \geq 2$ and $f \in BV_k[a, b]$, then it follows from Theorem 4 of [3] that f must be continuous.

Theorem 2 Let p be an integer such that $1 \leq p < k$. If $f \in BV_k^*[a, b]$, then $f \in BV_p^*[a, b]$, and

$$V_p(f) \leq p(p + 1) \ldots (k - 1)(b - a)^{k-p} V_k(f),$$

or

$$\bar{V}_p(f) \leq (b - a)^{k-p} \bar{V}_k(f).$$

With these results it is now possible to quote the required results to show that $BV_k^*[a, b]$ is a commutative Banach algebra.

<u>Theorem 3</u> If $f, g \in BV_k^*[a, b]$, then $fg \in BV_k^*[a, b]$,

and

$$V_k(fg) \leq 2^{k-1}(b - a)^{k-1}(k - 1)!V_k(f)V_k(g). \tag{3}$$

Finally, by putting

$$\|f\|_k^* = \alpha_k V_k(f),$$

where $\alpha_k = 2^{k-1}(b - a)^{k-1}(k - 1)!$, (3) becomes

$$\|fg\|_k^* \leq \|f\|_k^* \|g\|_k^*, \text{ as required.}$$

The details of the last two theorems appear in [5].

References

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