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DERIVATION BASES AND THE HAUSDORFF MEASURE

The purpose of this talk is to answer a question posed by Thomson [5, p.164] on the relation, if any, between the Hausdorff measure and the D derivation basis. It is shown that the Hausdorff measure equals the measure generated by the D derivation basis (1) when the derivative of h at 0 exists and is finite, (2) when the set is countable or (3) when the sum $\Sigma h(|I_n|)$ over the contiguous intervals of a given closed set converges. However, it is shown that the symmetric derivation basis is finite on more sets of finite Hausdorff measure then the measure from the D derivation basis.

When the lower right derivate of h at 0 is finite, the Hausdorff measure is a multiple of the Lebesgue measure. $\underline{D}^+h(0)$ is the multiple. When the upper right derivate of h at 0 is finite, $(h\circ m)_s(E)$, $(h\circ m)_D(E)$ and $(h\circ m)_{D^{\#}}(E)$ are a multiple of the Lebesgue measure and that multiple is $\overline{D}^+h(0)$.

The remaining question as to the relation between the measures is when h has an infinite derivate at 0. It is answered as follows: For the $D^{\#}$ derivation basis, the answer is trivial. If E is any non-empty set in [a,b] then $(h\circ m)_{D^{\#}}(E) = \infty$. As was mentioned above, when the sum $\Sigma h(|I_n|)$ over the contiguous intervals of a given closed set converge, the Hausdorff measure [3] and the measure generated by the D derivation basis agree and are equal to zero. However, there is a set for which $\Sigma h(|I_n|) = \infty$ and the set has D derivation measure zero. Also a set E can be constructed with Hausdorff measure zero and $(h\circ m)_s(E) = \infty$. For sets of finite, nonzero Hausdorff measure, sufficient conditions for the symmetric derivation basis measure and D derivation basis measure to be infinite are $d_D(x) = \underline{\lim}_{|I|\to 0}\mu^h(E\cap I)/h(|I|)$ where x is a left hand endpoint of I, then $(h\circ m)_D(E) = \infty$ when $\int_E d_D^{-1}(x)\,d\mu^h(x) = \infty$.

Note that if E is the Cantor set and $h(x) = x^\alpha$ where $\alpha = \log 2/\log 3$, Besicovitch [1] proved that $d_D(x) = 0$ a.e. μ^h on E. Therefore $(h\circ m)_D(E) = \infty$. A set was constructed by Besicovitch [1] that has $d_s(x) = 0$ ($d_s(x)$ is defined similarly). Since an analogous theorem holds for the symmetric derivation basis measure, there exists a set of finite, non-zero Hausdorff measure which has infinite measure with respect to the symmetric derivation basis. A sufficient condition for $(h\circ m)_s$ to be finite is given by the following theorem: Let $E = \cup_n E_n$ where E and E_n are measurable sets and the E_n are pairwise disjoint. If $d_s(x) \geq d_n > 0$ for each $x \in E_n$ and $\Sigma_{n=1}^{\infty} d_n^{-1}\mu^h(E_n) < \infty$, then $(h\circ m)_s(E) < \infty$. (A similar theorem holds for $(h\circ m)_D$.) If E is the Cantor set, then $d_s(x) > 1/h(12)$ for all $x \in E$. Therefore $(h\circ m)_s(E) < \infty$. Furthermore the $(h\circ m)_s$ measure of the Cantor set is strictly larger than the Hausdorff measure.

This talk is based on a paper which is derived from part of my dissertation under the direction of Prof. James Foran.

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