Generalized Riemann Derivatives

by

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Let a_i, b_i $(i=1,\ldots,n)$ be real numbers such that the b_i's are distinct, $\sum_{i=1}^{n} a_i b_i^k = 0$ for $k=0,1,\ldots,r-1$ and $\sum_{i=1}^{n} a_i b_i^r = 1$. The generalized Riemann derivative (GRD) of order r of the real function f at the point x is defined by

$$D^r f(x) = \lim_{h\to 0} \frac{\sum_{i=1}^{n} a_i f(x+b_i h)}{h^r/r!} .$$

Replacing the limit by limsup and liminf, we obtain the bilateral upper and lower GRD's $\overline{D}^r f(x)$ and $\underline{D}^r f(x)$, respectively. The unilateral GRD's $\overline{D}^r_+ f(x)$, etc. are defined by restricting h to be positive or negative. Obviously, this notion of derivative depends on the choice of the numbers a_i, b_i. However, it is easy to see that if f has a finite ordinary r^{th} derivative, or, more generally, a finite r^{th} Peano derivative at x then $D^r f(x) = f_{(r)}(x)$, independently of the choice of a_i and b_i. These derivatives, which contain the symmetric and Riemann-Schwarz derivatives as special cases, were investigated by M. Ash in his thesis and subsequent papers. In [1] and [2] Ash poses several problems concerning these derivatives and points to possible applications in multiple trigonometrical series and numerical analysis.

In this paper we investigate the possible monotonicity and convexity theorems in terms of GRD's of order 1 and 2. The general question for $r=1$ is to decide whether the condition $\underline{D}^1_+ f \geqq 0$, for a continuous function f, implies the monotonicity of f. Let $f:[a,b]\to\mathbb{R}$ be continuous. We denote by G^1_f the union of all open intervals on which f is increasing, and put $F^1_f = [a,b]\setminus G^1_f$.

<u>Theorem 1</u>. Let f be continuous on $[a,b]$ and let D^1 be an n-term GRD of order 1. If $\underline{D}_+ f(x) \geqq 0$ for every $x\in(a,b)$ then F^1_f is nowhere dense and $\underline{f}'_+(x) = -\infty$ holds on a residual subset of F^1_f.

<u>Corollary</u>. Let f be continuous on $[a,b]$ and let D^1 be an n-term GRD of order 1. If $\underline{D}^1_+ f \geqq 0$ and $\underline{f}'_+ > -\infty$ for every $x\in(a,b)$ then f is increasing.

<u>Theorem 2</u>. Let D^1 be a three-term GRD of order 1 with $b_1 < b_2 < b_3$.

(a) Suppose either

$b_1 < 0 < b_2 < b_3$ and $a_1 > 0,\ a_2 < 0,\ a_3 > 0$,

or

$b_1 < b_2 < 0 < b_3$ and $a_1 < 0,\ a_2 > 0,\ a_3 < 0$.

Then there is a nonconstant continuous decreasing function $f:[a,b]\to\mathbb{R}$ such that $D^1_+f(x)\geqq 0$ for every $x\in(a,b)$.

(b) In all other cases, if $f:[a,b]\to\mathbb{R}$ is continuous and $\underline{D}^1_+f(x)\geqq 0$ for every $x\in(a,b)$ then f is increasing.

(For example, the system $a_1=1$, $a_2=-2$, $a_3=1$, $b_1=-1$, $b_2=1$, $b_3=4$ belongs to case (a) and hence there is no monotonicity theorem for the corresponding GRD $\underline{D}^1_+$. On the other hand, for the system $a_1=1, a_2=-2$, $a_3=1$, $b_1=-2$, $b_2=-1$, $b_3=1$ there is a monotonicity theorem.)

Now we turn to the convexity theorems. Let $f:[a,b]\to\mathbb{R}$ be continuous. We denote by G^2_f the union of all relatively open intervals on which f is convex and put $F^2_f=[a,b]\setminus G^2_f$.

Theorem 3. Suppose $f:[a,b]\to\mathbb{R}$ is continuous and D^2 is an arbitrary n-term GRD of order 2. If $\underline{D}^2_+f(x)\geqq 0$ for every $x\in(a,b)$ then F^2_f is nowhere dense.

Theorem 4. Let D^2 be an n-term GRD of order 2 with $b_i\geqq 0$ $(i=1,\ldots,n)$. Then for every nowhere dense closed subset F of $[a,b]$ with neither a nor b an isolated point of F, there exists a continuous function $f:[a,b]\to\mathbb{R}$ such that $\underline{D}^2_+f(x)\geqq 0$ for every $x\in(a,b)$ and $F^2_f=F$.

As the previous theorem shows, the unilateral condition

$\underline{D}_{+}^{2}f \geq 0$ in general does not imply the convexity of f. In the next two theorems we investigate the consequences of this unilateral condition for three-term derivatives.

<u>Theorem 5</u>. Let D^2 be a three-term GRD of order 2 with $b_1 < b_2 = 0 < b_3$. If $f: [a,b] \to \mathbb{R}$ is continuous and $\underline{D}_{+}^{2}f(x) \geq 0$ for every $x \in (a,b)$ then f is convex.

<u>Theorem 6</u>. Let D^2 be a three-term GRD of order 2 with $b_1 < 0 < b_2 < b_3$. Suppose that $f: [a,b] \to \mathbb{R}$ is continuous, $\underline{D}_{+}^{2}f(x) \geq 0$ for every $x \in (a,b)$ and f is not convex. Then there exists a point $d \in (a,b)$ such that f is convex in a left neighbourhood of d and $f'_{+}(d) = -\infty$.

<u>Corollary</u>. Let D^2 be a three-term GRD of order 2 with $b_1 < 0 < b_2 < b_3$. Let $f: [a,b] \to \mathbb{R}$ be continuous and suppose that $\underline{D}_{+}^{2}f(x) \geq 0$ and $\underline{f}'_{+}(x) > -\infty$ hold for every $x \in (a,b)$. Then f is convex.

Now we turn to the bilateral condition $\underline{D}^{2}f \geq 0$ with three-term GRD's. We show that this condition implies that F_f^2 has many isolated points, and if the b_i's are of the same sign, we cannot say more. On the other hand, if $b_1 < 0 < b_3$ then sometimes we have a convexity theorem, depending on the location of b_2.

Theorem 7. Let D^2 be a three-term GRD of order 2 and let $f:[a,b]\to\mathbb{R}$ be continuous. If $\underline{D}^2f(x)\geqq 0$ for every $x\in(a,b)$ then the set of isolated points of F_f^2 is dense in F_f^2.

Theorem 8. Let D^2 be an n-term GRD of order 2 with $b_i\geqq 0$ $(i=1,\dots,n)$. Let F be a closed subset of $[a,b]$ such that neither a nor b is an isolated point of F and the set of isolated points of F is dense in F. Then there exists a continuous function $f:[a,b]\to\mathbb{R}$ such that $\underline{D}^2f(x)\geqq 0$ for every $x\in(a,b)$ and $F_f^2=F$.

Theorem 9. Let D^2 be a three-term GRD of order 2 with $b_1<0\leqq b_2<b_3$. Suppose that either $b_2=0$ or

$$\frac{\ln|b_2/b_1|}{b_2-b_1}\leqq\frac{\ln|b_3/b_1|}{b_3-b_1}.$$

If $f:[a,b]\to\mathbb{R}$ is continuous and $\underline{D}^2f(x)\geqq 0$ for every $x\in(a,b)$ then f is convex.

Theorem 10. Let D^2 be a three-term GRD of order 2 such that $b_1<0<b_2<b_3$ and

$$\frac{\ln|b_2/b_1|}{b_2-b_1}>\frac{\ln|b_3/b_1|}{b_3-b_1}.$$

Let F be a nowhere dense closed subset of $[a,b]$ such that neither a nor b is an isolated point of F and the set of isolated points of F is dense in F. Then there exists a continuous function $f:[a,b]\to\mathbb{R}$ such that $\underline{D}^2f(x)\geq 0$ for every $x\in(a,b)$ and $F_f^2=F$.

Our next result answers a question posed by Ash in [1].

<u>Theorem 11.</u> Let D^2 be a three-term GRD of order 2 with $b_1 < 0 < b_3$. If $f:[a,b]\to \mathbb{R}$ is continuous and $D_+^2 f(x) = 0$ for every $x \in (a,b)$ then f is linear.

<u>References</u>

[1] J.M. Ash, Very generalized Riemann derivatives, R.A.Ex. 11(1985-86), 10-29.

[2] J.M Ash, Generalized differentiation and summability, to appear in R.A.Ex.

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