Limits under the integral sign

by Ralph Henstock

Many parts of the calculus need the interchange of limit and integral, for example, the continuity and differentiability of an integral with respect to a parameter, an infinite series of integrals, and the exchange of order of integration in a repeated integral. As we can now define Lebesgue and even Denjoy-Perron integrals by using Riemann sums, it is possible to find necessary and sufficient conditions for the property

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx,$$

given that $f_n(x)$ tends to a finite limit $f(x)$ almost everywhere, and that each $f_n(x)$ is integrable over $[a,b]$.

Assuming generalized Riemann integration theory, the necessary and sufficient conditions are that

(1) $(E)\sum f_{m(x)}(x)(v-u)\in C$

for some compact set C of arbitrarily small diameter, some finite positive function $M(x)$ on $[a,b]$, all positive integer valued functions $m(x)\geqslant M(x)$ on $[a,b]$, some function $\delta(x)>0$ on $[a,b]$, and all δ-fine divisions E of $[a,b)$;

given $\varepsilon>0$, there are a number F, an integer $N>0$, and a function $\delta_n(x)>0$ on $[a,b]$ and depending on n, with

(2) $F-\varepsilon<(E)\sum f_n(x)(v-u)<F+\varepsilon$

for all δ_n-fine divisions E of $[a,b)$ and all $n\geqslant N$.

For the property

$$\frac{d}{dy}\int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f(x,y)}{\partial y}\,dx,$$

and given the integrability in [a,b) of f(x,y) for each fixed y in a neighbourhood of y=c, the necessary and sufficient conditions are that

(3) $(E)\Sigma\{f(x,m(x))-f(x,c)\}(v-u)/\{m(x)-c\}\in C$

for some compact set C of arbitrarily small diameter, some positive function M on [a,b], all functions m satisfying $c-M(x)<m(x)<c+M(x)$, $m(x)\neq c$, in [a,b], some function $\delta>0$ on [a,b], and all δ-fine divisions E; and, given $\varepsilon>0$, there are an $N>0$, a number F, and a function $\delta_y(x)>0$ on [a,b] depending on $y\neq c$, with

(4) $(F-\varepsilon)|y-c|<(E)\Sigma\{f(x,y)-f(x,c)\}(v-u)\,\mathrm{sgn}(y-c)$
$<(F+\varepsilon)|y-c|$

for all δ_y-fine divisions E of [a,b) and all y in $0<|y-c|<N$.

For the property

$$\int_A^B\left\{\int_a^b g(x,y)\,dx\right\}dy = \int_a^b\left\{\int_A^B g(x,y)\,dy\right\}dx$$

given the integrability of g(x,y) with respect to each variable, keeping the other fixed at an arbitrary point of its range, the necessary and sufficient conditions are that

(5) $(E_x)\Sigma\{(E_{xy})\Sigma g(x,y)(w-t)\}(v-u)\in C$

for some compact set C of arbitrarily small diameter, some function $\delta(x)>0$ on [a,b], some function $\delta_{xy}(y)>0$ on [A,B] for each $x\in[a,b]$, all δ-fine divisions E_x of [a,b), and all δ_{xy}-fine divisions E_{xy} of [A,B];

given $\varepsilon>0$, there are a number F and a function $\delta(y)>0$ on [A,B] such that for all δ-fine divisions E_y of [A,B), a function $\delta_{yx}(x)>0$ on [a,b] depending on E_y, and all δ_{yx}-fine divisions E_{yx} of [a,b),

(6) $F-\varepsilon<(E_y)\Sigma\{(E_{yx})\Sigma g(x,y)(v-u)\}(w-t)<F+\varepsilon.$

Interchanging x and y then (5), (6) would be interchanged. Finally we generalize the Carathéodory theory of ordinary differential equations.

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