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ITERATES FOR A RESIDUAL CLASS OF FUNCTIONS

The talk was a report of joint work with A. M. Bruckner.

A considerable amount of recent research has been devoted to studying the iterative behavior of continuous functions mapping an interval into itself. Much of this work focuses on well-behaved functions; that is functions that satisfy certain differentiability conditions, are piecewise monotonic, and possess other properties that help in classifying iterative behavior.

These well-behaved functions have commonly been used as models for various physical, social and biological phenomena. In [P] (pg. 100, Theorem 5.2) one finds that all functions in a certain class that is sufficiently large to contain many of the functions that appear in practice, are of one of three types: i) a single periodic orbit attracts the orbits of most points; ii) a Cantor set attracts the orbits of most points; iii) there is sensitive dependence on initial conditions, i.e. for each x in the interval there exists $\varepsilon > 0$ such that for each $\delta > 0$, there exists a natural number n such that $|f^k(x-\delta,x+\delta)| > \varepsilon$ for all $k \geq n$. ($|J|$ denotes the length of the interval J.) Possibility (iii) implies very little attraction of orbits, but does allow certain chaotic behavior such as the existence of points whose orbits are dense. One also finds in the recent literature constructions of continuous functions that exhibit various sorts of chaotic behavior. See, for example, [BH], [K], [S].

Now, in the sense of Baire category, "typical" continuous functions do not satisfy any of the properties often assumed for well-behaved functions. (A property of a continuous function is considered typical if the property is shared by all continuous functions in a residual subset of the Banach space of continuous functions on a compact interval, with the sup norm.) It is natural to ask what errors might result if a well-behaved function were used to model a phenomenon when the true function describing the phenomenon is more typical. One is then led to study the dynamical structure of the iterates of a typical continuous function.

Our results use the following notation and terminology: C_0^1 denotes the Banach space of continuous functions which map the interval [0,1] into itself (with the sup norm). For a given $f \in C_0^1$, the orbit of a point $x \in [0,1]$, denoted $O(x)$, is the sequence $x, f(x), f \circ f(x), \ldots$.

We say that the orbit of x is attracted to a set H if H is the cluster set of $O(x)$.

We were able to obtain the following:

Theorem: There exists a residual subset of C_0^1 each of whose members f has the following properties:

i) To each x in some residual subset of [0,1] corresponds a Cantor set H such that the orbit of x is attracted to H.

ii) There are c pairwise disjoint such Cantor sets.

iii) If H is such a Cantor set, then f maps H homeomorphically onto itself and each x in H has a dense orbit in H.

iv) For each such attracting Cantor set H, the set $H^* \equiv \{x : \text{the cluster set of the orbit of } x \text{ is contained in } H\}$ is nowhere dense.

As consequences of this theorem (and its proof) we were able to show:

1) The typical continuous function has no stable periodic point (a periodic point with an interval of points around it, all of whose orbits are attracted to the same period). However we were able to show that periodic points for typical functions do exhibit a type of stability: in any neighborhood of a periodic point of period n, there is an interval which is mapped into itself after n iterates.

2) The typical continuous function has no points whose orbits are dense.

These results contrast with the situation for the well-behaved functions, which can have stable periodic points or points with dense orbits.

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