# Generalized Integrals in the Theory of Trigonometric, Haar, and Walsh Series.

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In the first lecture we survey some work done by Soviet mathematicians on the application of generalized integrals to trigonometric series.

The first group of these integrals was introduced to integrate sums of everywhere convergent trigonometric series, and to calculate the coefficients of those series by the Fourier formulae. They are the $T_{2s}$-, $P^2$-, SCP-, and MZ- integrals; see [2, 3, 4, 5].

We give results concerning the properties of the primitives of these integrals, see [6, 7, 8, 9], and their relations to the Denjoy integrals in the restricted sense and in the wide sense, see [10, 11, 12].

Among such properties we draw attention to the fact that for these integrals the first order primitive does not necessarily have the Luzin N-property, [10]. Furthermore an example of a function with such a primitive can be chosen from the sums of everywhere convergent trigonometric series, [12]. For more references see [13].

Another group of integrals discussed includes A-, and B- integrals, which were introduced to integrate the conjugate function of any summable function. Most results concerning these integrals surveyed in the lecture have been published, see [14, 15, 16]. We mention here only the recently published result by Pannikov, [16], that the B- integral is included in the A- integral.

The second lecture concerns mainly the problem of calculating coefficients from the sum of Haar and Walsh series. There are several definitions of integrals wide enough to make any everywhere convergent Haar and Walsh series a Fourier series in the sense of these integrals, see [18, 19, 20].

The starting point for investigating the above is the observation that the partial sums of order $2^n$ of either a Haar or Walsh series can be expressed in the form

$$S_{2^n}(x) = \frac{F(b_n) - F(a_n)}{b_n - a_n}$$

where $\{[a_n, b_n]\}$ is a sequence of binary intervals converging to x, and F is the sum of the (term by term) integrated series, which converges at least on the dyadic rationals. So the problem of calculating the coefficients from the sums of Haar and Walsh series is practically the same as that of recovering the primitive from a derivative with respect to a binary sequence $\mathcal{N}$ of nets, [21].

A simple way of solving the latter problem is to introduce a Perron type integral $P_{\mathcal{N}}$, [20], for which upper and lower derivatives $\bar{D}_{\mathcal{N}}$, $\underline{D}_{\mathcal{N}}$ of major and minor functions are not ordinary upper and lower derivatives, but instead are taken with respect to the binary sequence of nets $\mathcal{N}$. In addition the continuity of these major and minor functions M,m is with respect to $\mathcal{N}$, which means that $M(b_n)-M(a_n)\to 0$ and $m(b_n)-m(a_n)\to 0$ where $\{[a_n, b_n]\}$ is as above.

Once of the peculiar features of these binary derivatives is that the condition $D_{\mathcal{N}}F(x) > -\infty$ everywhere for a continuous F does not imply that F is VBG as is well known to be the case for ordinary differentiation. This allows us to construct a function that is both $P_n$-, and Denjoy integrable in the wide sense, the indefinite $P_n$- integral of this function not being a VBG function. This shows, as a corollary, that these two general integrals are not compatible. However, it is true that these two integrals agree on the class of finite $D_n$- derivatives, which means than any Haar or Walsh series converging everywhere to function integrable in the wide sense of Denjoy is a Fourier series of its sum in the sense of this integral, [1].

A survey of results in this field prior to 1973 can be found in [22].

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