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EXTREME POINT MULTIFUNCTIONS AND A GENERALIZED RADON-NIKODYM THEOREM

By using a generalized version of the Radon-Nikodym theorem, we show that under suitable restrictions the bilinear integrals of a multifunction and the corresponding extreme point multifunction are equal.

1. INTRODUCTION

The integration of multifunctions has been studied extensively in recent years by numerous authors. The foundations were laid by R.J. Aumann [2], C. Castaing [6], K. Kuratowski and C. Ryll-Nardzewski [12], and others. C. Castaing [6] and C.J. Himmelberg and F.S. van Vleck [10] showed that under suitable restrictions the measurability of a multifunction F implies the measurability of the multifunction ext F, where ext $F(t)$ is the set of extreme points of $F(t)$. The main purpose of this paper is to show, by using a generalized theorem of Radon-Nikodym, that the bilinear integrals (in the sense of N. Dinculeanu [8]) of these two multifunctions are equal. This extends corresponding results on the same topic.

2. PRELIMINARIES

Throughout this paper T will denote a non-empty point set on which no topological structure is required. Let V be a Banach space and $\mathcal{C}$ a ring of subsets of T. Let $m: \mathcal{C} \to V$ be a measure. For every set $A \in \mathcal{C}$, let $|m|(A)$ be the *variation* of m on the set A. If $|m|(A) < \infty$ for every $A \in \mathcal{C}$, then $|m|$ is of finite variation with respect to $\mathcal{C}$. Extend the finite measure $|m|$ on $\mathcal{C}$ to a measure $|m|^*$ on the σ-algebra $\mathcal{P}(|m|)$ of all $|m|$*-measurable* sets. The class $\Sigma(|m|) = \{E \in \mathcal{P}(|m|) \mid |m|^*(E) < \infty\}$ is the δ-ring of all $|m|$-*integrable* sets. The restriction of $|m|^*$ to $\Sigma(|m|)$ is denoted by $|m|$. If $E \in \Sigma(|m|)$ and $|m|(E) = 0$, then E is called $|m|$-*negligible*.

2.1 <u>DEFINITION</u> ([8], p. 179). Denote by $C(|m|)$ the collection of all classes $\mathcal{A} = \{A_i | i \in I\}$ of disjoint $|m|$-integrable sets such that $T - \bigcup_{i \in I} A_i$ is $|m|$-negligible and such that for every set $A \in \mathcal{C}$ there exist an $|m|$-negligible set $N \subset A$ and an at most countable set $J \subset I$ with $A - N = \bigcup_{i \in J} (A \cap A_i)$. We say that the measure $|m|$ has the *direct sum property* if $C(|m|) \neq \emptyset$. A measure of finite variation is said to have the direct sum property if its variation has this property.

2.2 SOME PROPERTIES. (a) The measure m on $\mathcal{C}$ can be extended to a measure m on $\Sigma(|m|)$ (see [8], p. 76).
(b) If $\mathcal{C}$ is a σ-algebra and if $|m|$ is complete on $\mathcal{C}$, then $\mathcal{C} = \Sigma(|m|) = \mathcal{P}(|m|)$. (See Section 5).
(c) If T is a countable union of sets of $\mathcal{C}$, then $|m|^*$ is a σ-finite and complete measure on $\mathcal{P}(|m|)$. Thus $|m|$ on $\Sigma(|m|)$ is also complete.
(d) Whenever m is supposed to be non-atomic, it must be understood that m is non-atomic on $\Sigma(|m|)$, that is, the extended measure m in non-atomic. This convention is necessary, because the extension of a non-atomic measure need not be non-atomic, see [4], p. 2 or [21], p. 67 for examples.
(e) If m is non-atomic on $\Sigma(|m|)$, so is $|m|$.
(f) If T is a countable union of sets of the δ-ring $\Sigma(|m|)$, then m has the direct sum property. This follows from the fact that $\mathcal{C} \subset \Sigma(|m|)$.

Throughout the paper U will denote a Banach space. A function $f: T \to U$ is $|m|$*-measurable* if $f^{-1}(C) \in \mathcal{P}(|m|)$ for every closed set C in U. A multifunction $F: T \to U$ is a function whose domain is T and whose values are non-empty subsets of U. If $A \subset U$, then $F^-(A) = \{t \in T | F(t) \cap A \neq \emptyset\}$. A multifunction $F: T \to U$ is $|m|$-*measurable (weakly* $|m|$*-measurable)* if $F^-(A) \in \mathcal{P}(|m|)$ for every closed (open) subset A of U. A function $f: T \to U$ is called a *selector* for F if $f(t) \in F(t)$ $|m|$-a.e. on T. The set of all $|m|$-measurable selectors of F will be denoted by $\mathcal{S}_F$. If $f \in \mathcal{S}_F$, consider the equivalence class $\tilde{f} = \{g: T \to U | g(t) = f(t)\ |m|\text{-a.e. on } T\}$. Write $S_F = \{\tilde{f} | f \in \mathcal{S}_F\}$.

A multifunction $F: T \to U$ admits a *Castaing representation* if there exists a countable set $M = \{f_i | i \in I\} \subset S_F$ such that $M(t) = \{f_i(t) | i \in I\}$ is dense in $F(t)$ $|m|$-a.e. on T. (See [6], p. 116).
Let W be a third fixed Banach space and consider a bilinear transformation $(u,v) \to uv$, defined on $U \times V$ into W such that $\|(u,v)\| \leq \|u\| . \|v\|$.

The vector integral being employed is the "bilinear" or "m-integral" of Dinculeanu. Let

$$\mathcal{E}_U(\Sigma(|m|)) = \{f: T \to U | f = \sum_{i \in I} x_i \chi_{A_i},\ x_i \in U,\ A_i \in \Sigma(|m|) \text{ and } I \text{ is a finite index set}\}.$$

A function $f: T \to U$ is *m-integrable* if there exists a Cauchy sequence (f_n) in $\mathcal{E}_U(\Sigma(|m|))$ such that $f_n \to f$ $|m|$-a.e. on T. Then $\int f(t)dm \in W$.

The space of all m-integrable functions $f: T \to U$ will be denoted by $\mathcal{L}^1_U(m)$. The set of all $|m|$-integrable selectors of $F: T \to U$ will be denoted by I_F. Then $I_F \subset S_F$. Write $I_f = \{\tilde{f} | f \in I_F\}$. If $f \in \mathcal{L}^1_U(m)$ and $A \in P(|m|)$, then $f\chi_A \in \mathcal{L}^1_U(m)$ and $\int_A f(t)dm = \int f(t)\chi_A(t)dm$. If $A \in P(|m|)$, then the integral of a multifunction $F: T \to U$ over A is defined by

$$\int_A F(t)dm = \{\int_A f(t)dm | f \in I_F\}.$$

We observe that $\int_A F(t)dm$ exists, even if F is not $|m|$-measurable. Furthermore, $\int_A F(t)dm$ may be empty, even if $U = \mathbb{R}$.
A multifunction $F: T \to U$ is said to be *p-integrably bounded,* $1 \leq p < \infty$, if there exists a $k \in \mathcal{L}^p_{\mathbb{R}}(|m|)$ such that

$$\sup\{\|u\| \,|\, u \in F(t)\} \leq k(t) \quad |m|\text{-a.e. on } T.$$

If $F: T \to U$ is 1-integrably bounded by $k \in \mathcal{L}^1_{\mathbb{R}}(|m|)$, we say that F is integrably bounded by k.
Let P be a property possessed by some subsets of the Banach space U. A multifunction $F: T \to U$ is said to be *point*-P if for every $t \in T$, $F(t)$ has property P. Denote the topological dual of U by U^{*}. Following M. Valadier [19], we say that the point-compact convex multifunction $F: T \to U$ is *scalarwise* $|m|$*-measurable* (*-integrable*) if

for every $x' \in U'$, the function $h_{x'}: T \to \mathbb{R}$, defined by

(*) $h_{x'}(t) = \sup\{\langle x,x'\rangle \mid x \in F(t)\}$

is $|m|$-measurable (-integrable).

2.3 <u>DEFINITION</u>. If X is a Banach space and Z a subspace of X', then Z is said to be a *norming subspace* of X' if

$$\|x\| = \sup\left\{\frac{|\langle x,z\rangle|}{\|z\|} \;\middle|\; z \in Z,\ z \neq 0\right\}, \quad \text{for every } x \in X.$$

Then, X can be imbedded isometrically in Z'.

If X and Y are linear spaces, then the space of all linear transformations from X to Y will be denoted by $L^*(X,Y)$.

2.4 <u>DEFINITION</u>. Let X and Y be Banach spaces. We say that a function $U: T \to L^*(X,Y)$ is *simply* $|m|$*-measurable*, if for every $x \in X$ the function $\phi_x: T \to Y$, defined by $\phi_x(t) = U(t)x$, is $|m|$-measurable.

2.5 <u>DEFINITION</u>. Let X and Y be Banach spaces and $Z \subset Y'$ a norming subspace. We say that a function $U: T \to L^*(X,Y)$ is Z*-weakly* $|m|$*-measurable*, if for every $x \in X$ and every $z \in Z$, the function $\phi_{x,z}: T \to \mathbb{R}$, defined by $\phi_{x,z}(t) = \langle U(t)x,z\rangle$, is $|m|$-measurable.
Denote by $\mathcal{B}_U$ the Borel σ-algebra of U and by $T(P(|m|) \times \mathcal{B}_U)$ the σ-algebra generated by the class

$$P(|m|) \times \mathcal{B}_U = \{A \times B \mid A \in P(|m|),\ B \in \mathcal{B}_U\}.$$

The graph of the multifunction $F: T \to U$ is the set

$$G(F) = \{(t,u) \in T \times U \mid u \in F(t)\}.$$

A topological space is *Polish* if it is separable and metrizable by a complete metric; it is *Suslin* if it is metrizable and the continuous image of a Polish space.
If $F: T \to U$ is a multifunction, then the multifunction $\text{ext}\, F: T \to U$, defined for every $t \in T$ by

$$(\text{ext}\, F)(t) = \{u \in F(t) \mid u \text{ is an extreme point of } F(t)\},$$

is called the *extreme point multifunction* determined by F. Multifunctions will be denoted by the capitals F, G and H. If $A \subset U$, then co A denotes the *convex hull* of A.

3. SOME BASIC RESULTS

We state the following propositions in forms which are adequate for the sequel.

3.1 PROPOSITION ([12], p. 398). *Let U be separable and $F: T \to U$ point-closed and weakly $|m|$-measurable. Then F has an $|m|$-measurable selector.*

3.2 COROLLARY. *Let U be separable and $F: T \to U$ point-closed and $|m|$-measurable. Then F has an $|m|$-measurable selector.*

PROOF. If 0 is open in U, then $0 = \bigcup_{n=1}^{\infty} C_n$, where the C_n are all closed in U. Then $F^{-}(0) = \bigcup_{n=1}^{\infty} F^{-}(C_n) \in \mathcal{P}(|m|)$. Thus, F is weakly $|m|$-measurable and proposition 3.1 holds. ∇

3.3 PROPOSITION ([20], p. 868). *Let T be a countable union of sets of the ring $\mathcal{C}$, U separable and $F: T \to U$ point-closed. Then the following conditions are equivalent:*

(1) F *is* $|m|$-*measurable*;

(2) F *is weakly* $|m|$-*measurable*;

(3) $\mathcal{G}(F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$;

(4) F *admits a Castaing representation*.

Note that the assumption on T implies completeness of the measure space $(T, \mathcal{P}(|m|), |m|^*)$, see 2.2(c). This in turn implies that $\mathcal{P}(|m|)$ is a Suslin family (see [18], p. 50 or [20], p. 864), as is required for proposition 3.3 to hold. It is possible to show by means of a suitable example that the completeness of $(T, \mathcal{P}(|m|), |m|^*)$ is indeed necessary, see for example [1], p. 27. A further requirement in [20], p. 868 is that U be Suslin, which it surely is since it is Polish. These remarks also apply to the proposition below, originally proved for a complete measurable space, a Suslin space U and where F need neither be closed-valued nor $|m|$-measurable. This proposition is a generalization of the so-called Von Neumann-Aumann selection theorem, see [2] or [14], p. 69.

3.4 PROPOSITION ([17], p. 7.11). Let T be a countable union of sets of $\mathcal{C}$, U separable and $F: T \to U$ such that $G(F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$. Then F has an $|m|$-measurable selector.

3.5 PROPOSITION. If $F: T \to U$ is point-compact convex and $|m|$-measurable, then F is scalarwise $|m|$-measurable.

PROOF. The function $h_{x'}: T \to \mathbb{R}$ defined in (*) is $|m|$-measurable, see [7], lemma 5, p. 231. Consequently, F is scalarwise $|m|$-measurable. ∇

3.6 PROPOSITION ([9], p. 439). A non-emtpy compact subset of a locally convex linear topological Hausdorff space has extreme points.

We now employ a theorem of M. Benamara [3] which deals with

(i) a point-compact convex $F: T \to U'$ which is scalarwise $|m|$-measurable, i.e. if for every $x \in U$, the function $h_x: T \to \mathbb{R}$, defined by

$$h_x(t) = \sup\{ \langle x',x \rangle \mid x' \in F(t)\}$$

is $|m|$-measurable;

(ii) a complete measure space.

With remark 2.2(c) in mind, we now have:

3.7 PROPOSITION ([3], p. 1249). Let T be a countable union of sets of the ring $\mathcal{C}$, U separable and $F: T \to U'$ point-$\sigma(U',U)$-compact convex and scalarwise $|m|$-measurable. Then the set ext S_F of all extreme points of S_F is non-empty and equal to the set $S_{\text{ext } F}$.

3.8 PROPOSITION ([10], p. 725). If $F: T \to \mathbb{R}^n$ is point-compact convex and $|m|$-measurable, then $G(\text{ext } F) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_{\mathbb{R}^n})$. Furthermore, if T is a countable union of sets of the ring $\mathcal{C}$, then ext F is $|m|$-measurable.

3.9 PROPOSITION. Let (F_n) be a sequence of multifunctions, $F_n : T \to U$, with $\mathcal{G}(F_n) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U)$ for all n. Define the multifunctions $G_i : T \to U$, $i = 1,2,3,4$ by the respective equalities $G_1(t) = \bigcup_{n=0}^{\infty} F_n(t)$; $G_2(t) = \bigcap_{n=0}^{\infty} F_n(t)$; $G_3(t) = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} F_k(t)$ and

$G_4(t) = \bigcap_{n=0}^{\infty} \bigcup_{k=n}^{\infty} F_k(t)$. Then we have ıt

$$\mathcal{G}(G_i) \in \mathcal{T}(\mathcal{P}(|m|) \times \mathcal{B}_U), \quad i = 1,2,3,4.$$

PROOF. Routine.

3.10 PROPOSITION ([16], pp. 166, 167). If $\dim U < \infty$, then a compact convex subset A of U equals the convex hull of the set of its extreme points, in symbols $A = \text{co ext } A$.

The two propositions below are stated in general terms.

3.11 PROPOSITION. Let X and Y be linear spaces. If $f \in L^*(X,Y)$ and if C is a non-empty convex subset of X and B an extreme subset of $f(C)$, then $f^{-1}(B) \cap C$ is an extreme subset of C.

3.12 PROPOSITION (M. Krein and D. Milman [11]). If A is a compact subset of a locally convex linear topological Hausdorff space and E is the set of extreme points of A, then $A \subset \overline{co}\, E$, where $\overline{co}\, E$ denotes the closure of the convex hull of E. Consequently, $\overline{co}\, A = \overline{co}\, E$. If, in addition, A is convex, then each closed extreme subset of A contains an extreme point of A and $A = \overline{co}\, E$.

4. MAIN RESULTS

4.1 THEOREM. If U is separable and $F: T \to U$ is integrably bounded, point-closed and $|m|$-measurable, then $\int_A F(t)dm \neq \emptyset$ for every $A \in \mathcal{P}(|m|)$.

PROOF. Corollary 3.2 asserts that F has an $|m|$-measurable selector

f. If $k \in \mathcal{L}^1_{\mathbb{R}}(|m|)$ is the bounding function, then $\|f(t)\| \leqq k(t)$ $|m|$-a.e., hence, $f \in \mathcal{L}^1_U(m)$. Consequently, $f \in I_F$, and so $\int_A F(t)dm \neq \emptyset$ for every $A \in P(|m|)$. ∇

D. Blackwell [5] extended Lyapunov's convexity theorem by proving that the ranges of certain vector integrals with values in $\mathbb{R}^n$ are compact and, in the non-atomic case, convex. The convexity part of Blackwell's theorem was generalized by H. Richter [15]. By keeping proposition 3.12 in mind, we state Richter's theorem in the following form:

4.2 THEOREM ([15], p. 86). (1) If $F: T \to \mathbb{R}^n$ and m is non-atomic, then $\int_A F(t)d|m|$ is convex for every $A \in \Sigma(|m|)$.
(2) Let T be a countable union of sets of C, m non-atomic and $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-measurable. Then $\int_A F(t)d|m|$ is compact and convex for every $A \in \Sigma(|m|)$.

The detailed proofs of theorems 4.3, 4.6 and 4.7 can be found in [13].

4.3 THEOREM. Let T be a countable union of sets of C, m non-atomic and $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-measurable. Then

$$\int_A F(t)d|m| = \int_A (\text{ext } F)(t)d|m|$$

for every $A \in \Sigma(|m|)$.

4.4 THEOREM ([8], p. 263). If $m: C \to V \subset L(U,W)$ has the direct sum property and Z is a norming subspace of W', then there exists a function $U_m: T \to L(U,Z')$ having, among others, the following properties:
(1) $\|U_m(t)\| = 1$ $|m|$-a.e. on T;
(2) $< U_m f, z >$ is $|m|$-integrable, and

$$< \int f(t)dm, z > = \int < U_m(t)f(t), z > d|m|,$$

for $f \in \mathcal{L}^1_U(m)$ and $z \in Z$;
(3) We can choose $U_m(t) \in L(U,W)$ for every $t \in T$ in the case that $W = Z'$.

4.5 REMARKS. (a) In the proof of theorem 4.4, the function U_m is defined in such a way that for every $u \in U$ and for every $z \in Z$, the function $\phi_{u,z}: T \to \mathbb{R}$, defined by $\phi_{u,z}(t) = < U_m(t)u, z >$, is locally $|m|$-integrable, that is, $\phi_{u,z}\chi_A$ is $|m|$-integrable for every set $A \in C$, see [8], p. 163, definition 1. Then $\phi_{u,z}\chi_A$ is $|m|$-measurable for every set $A \in C$. By [8], p. 100, corollary, $\phi_{u,z}\chi_A$ is $|m|$-measurable.

(b) Suppose now that $W = Z'$. Then, by theorem 4.4(3), we have that $U_m: T \to L(U,W)$. Definition 2.5 and (a) above then show that U_m is Z-weakly $|m|$-measurable. Suppose further that Z', and hence W, is separable. Then U_m is simply $|m|$-measurable, see [8], p. 105, proposition 22. If now $f: T \to U$ is $|m|$-measurable, then the function $g: T \to Z' = W$, defined by $g(t) = U_m(t)f(t)$, is $|m|$-measurable, see [8], p. 102, proposition 16. By theorem 4.4(1), we now have that

$$\|U_m(t)f(t)\| \leq \|U_m(t)\|.\|f(t)\| = \|f(t)\| \quad |m|\text{-a.e. on } T.$$

If $f \in \mathcal{L}^1_U(m)$, then $U_m f \in \mathcal{L}^1_W(|m|)$. Under the conditions sketched above and from theorem 4.4(2) we obtain, for $f \in \mathcal{L}^1_U(m)$ and every $z \in Z$, that

$$< \int f(t)dm, z > = \int < U_m(t)f(t), z > d|m|$$
$$= < \int U_m(t)f(t)d|m|, z >.$$

The second equality above follows from [8], p. 123, corollary to the proposition 7. We then have that

$$\int f(t)dm = \int U_m(t)f(t)d|m|.$$

4.6 THEOREM. <u>Let T be a countable union of sets of the ring C, U separable and F: T → U integrably bounded, point-compact and $|m|$-measurable. If W is separable, $W = Z'$ where Z is a norming subspace of W', $m: C \to V \subset L(U,W)$ and $U_m: T \to L(U,Z') = L(U,W)$ is the function whose existence is guaranteed by theorem 4.4, then</u>

$$\int_A F(t)dm = \int_A U_m(t)F(t)d|m|, \quad \text{for every } A \in P(|m|).$$

4.7 THEOREM. <u>Let T be a countable union of sets of the ring C, $F: T \to \mathbb{R}^n$ integrably bounded, point-compact convex and $|m|$-mea-</u>

surable and let $m: \Sigma(|m|) \to \mathbb{R}^p$ be non-atomic. Then

$$\int_A F(t)\,dm = \int_A (\text{ext } F)(t)\,dm, \quad \text{for every } A \in \Sigma(|m|).$$

5. EXAMPLES

The main purpose of this section is to show by means of illustrative examples that parts of the hypotheses of theorems 4.3 and 4.7 cannot be weakened.

5.1 EXAMPLE. Let $T = \{t_o\}$, $\Sigma = \{\emptyset, T\}$ and $m: \Sigma \to \mathbb{R}$ be defined by $m(T) = 1$, $m(\emptyset) = 0$. Then m is an atomic measure and $m = |m|$. Define $F: T \to \mathbb{R}$ by $F(t) = [1,2]$. Then F satisfies the conditions of theorems 4.3 and 4.7. Furthermore,

$$(\text{ext } F)(t) = \{1,2\} = \int (\text{ext } F)(t)\,dm.$$

If $f: T \to \mathbb{R}$ is defined by $f(t) = 1\frac{1}{2}$, then $f \in I_F$ and $\int f(t)\,dm = 1\frac{1}{2} \in \int F(t)\,dm$. Thus,

$$\int F(t)\,dm \neq \int (\text{ext } F)(t)\,dm.$$

5.2 EXAMPLES. Let $T = [0,1]$, Σ be the Lebesgue σ-algebra of subsets of T and m the Lebesgue measure on T. Then m is non-atomic and $m = |m|$. (a) Define $F: T \to \mathbb{R}$ by $F(t) = \mathbb{R}$ for all $t \in T$. Then F is point-convex, but neither integrably bounded nor point-compact. Clearly, $(\text{ext } F)(t) = \emptyset$ for all $t \in T$ and

$$\int (\text{ext } F)(t)\,dm = \emptyset \neq \int F(t)\,dm.$$

(b) Define $F: T \to \mathbb{R}$ by $F(t) = (0,1)$ for all $t \in T$. Then F is integrably bounded and point-convex but not point-compact. As in (a) above we have that

$$\int (\text{ext } F)(t)\,dm = \emptyset \neq \int F(t)\,dm.$$

5.3 EXAMPLE. The space c_o is the Banach space of all sequences $x = (x_n)$ converging to zero. The space c_o is infinite dimensional and the closed unit ball A of c_o is non-compact and convex. Let T, Σ and m be as in 5.2 and consider $c_o = L(\mathbb{R}, c_o)$. Define $F: T \to c_o$ by $F(t) = A$ for all $t \in T$. Then F is clearly $|m|$-

measurable and integrably bounded. Since ext A = $\emptyset$ we have that

$$\int(\text{ext } F)(t)dm = \emptyset \neq \int F(t)dm.$$

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