Daniel Waterman, Department of Mathematics
Syracuse University, Syracuse, NY 13244-1150, U.S.A.

Recent Developments in Fourier Analysis and Generalized Bounded Variation

Bounded variation and its generalizations are associated with many aspects of Fourier series. Here we discuss three recent results dealing with these aspects. The first involves a new notion of summability which arises naturally from the consideration of Fourier series of functions of Λ-bounded variation. The second is concerned with functions whose Fourier series have small gaps and the significance of the assumption that such a function is of generalized bounded variation on a subinterval. The third result is an improvement of the Bohr-Pál theorem: we have shown that if f is continuous and 2π - periodic, then there is a homeomorphism of $[0, 2\pi]$ onto itself such that $f \circ g$ is the conjugate of a function of bounded variation.

1. SUMMABILITY OF FOURIER SERIES.

Let $\Lambda = \{\lambda_k\}$ be a nondecreasing sequence of real numbers with $\lambda_1 = 1$ and $\sum_1^\infty 1/k\lambda_k < \infty$. For a given positive integer n, let $I_{k,n} = ((k-1)\pi/n, k\pi/n]$, $k = 1, 2, \ldots, n$. Set

$$H_n(t) = \sum_1^n (k/\lambda_k)\, \chi_{I_{k,n}}(t)\, D_n(t)$$

where χ_E denotes the characteristic function of a set E and $D_n(t)$ is the

Dirichlet kernel,

$$D_n(t) = \frac{1}{2} + \sum_{1}^{n} \cos kt.$$

Extend $H_n(t)$ as an even, 2π - periodic function continuous at $t = 0$. There exist c and c' so that

$$c_n = \int_{-\pi}^{\pi} H_n(t)\, dt$$

satisfies

$$0 < c < c_n < c' < \infty.$$

We set $K_n(t) = H_n(t)/c_n$.

If f is an integrable function on the circle group T, we say that the Fourier series of f, $S(f)$ is (W, Λ) - summable to sum σ at x, i.e., $(W, \Lambda)\ S(f)(x) = \sigma$, if

$$\sigma_n(f, x) = \int_{-\pi}^{\pi} f(x + t)\, K_n(t)\, dt \longrightarrow \sigma \text{ as } n \longrightarrow \infty.$$

It is not difficult to show that replacing $D_n(t)$ by $(\sin nt)/t$ in the definition of the kernel yields an equivalent summability method.

The existence and properties of these methods were indicated briefly by Waterman [9]. Details will appear in a paper of D'Antonio and Waterman [2].

If $\sum_{1}^{\infty} 1/\lambda_k$ converges, we can show that $K_n(t)$ has the properties

(i) $\int_{-\pi}^{\pi} K_n(t)\, dt = 1,$

(ii) $\int_{-\pi}^{\pi} |K_n(t)|\, dt \leq C$ independent of n,

(iii) $\sup \{ |K_n(t)| : 0 < \delta \leq t \leq \pi \} \longrightarrow 0$ as $n \longrightarrow \infty$ for each δ.

The implications of these properties for a summability kernel are well-known [10, vol. I, chap. III]. For example, if $f(x \pm 0)$ are defined and finite, then

$$\sigma_n(f, x_0) \longrightarrow \frac{1}{2}\{f(x_0 + 0) + f(x_0 - 0)\}.$$

However, for this particular method, more can be shown, namely,

$\sigma_n(f, x) \longrightarrow f(x)$ at every Lebesgue point of f.

Let us now suppose that $\sum 1/\lambda_k = \infty$. $\{I_n\}$ will denote a collection of nonoverlapping intervals in T. If $I = [a,b]$, then $f(i) = f(b) - f(a)$. We say that f is of Λ-bounded variation (ΛBV) if $\sum_1^\infty |f(I_k)|/\lambda_k < \infty$ for every $\{I_n\}$. This is known to imply that the collection of sums $\sum |f(I_n)|/\lambda_n$ is bounded [8].

It can be shown that if $f \in \Lambda BV$ on $[0, 2\pi]$, then

$$(W, \Lambda)\ S(f)(x) = f(x)$$

for every x; summability is uniform on closed intervals of points of continuity. Further, if ΓBV is defined by a sequence $\Gamma = \{\gamma_k\}$ such that $\Gamma BV \setminus \Lambda BV \neq \emptyset$, then there is an $f \in \Lambda BV$ such that $S(f)$ is not (W, Λ) - summable at some point.

For the case $\Lambda = \{k\}$, (W, Λ) - summability is ordinary convergence and in that case we obtain a result of Waterman [7].

The results in the case $\sum 1/\lambda_k = \infty$ extend to a larger class of functions of generalized bounded variation, the class W_Λ.

Suppose $\{I_i\}$ is a finite collection of ordered contiguous intervals of length π/n and $\bigcup I_i$ is a closed interval in either $[x - \delta, x)$ or $(x, x + \delta]$, where $\delta > \pi/n$. If $I = [a, b]$, let $f(I) = f(b) - f(a)$.

Let

$$\Lambda V_n(f, x, \delta) = \sup \sum_1^n |f(I_i)|/\lambda_i,$$

the supremum being extended over all collections $\{I_i\}$ described above.

Let $\Lambda V(f, x, \delta) = \overline{\lim_{n\to\infty}} \Lambda V_n(f, x, \delta)$. We say that $f \in W_\Lambda$ if

$$\Lambda V(f, x, \delta) \longrightarrow 0 \text{ as } \delta \longrightarrow 0 \text{ for all } x.$$

It may be shown that $W_\Lambda \supset \Lambda BV$ properly.

These spaces were first discussed by Isaza [3] in the case $\Lambda = \{k\}$.

2. FOURIER SERIES WITH SMALL GAPS

A trigonometric series $\sum a_n \cos nx + b_n \sin nx = \sum A_n(x)$ is said to have "small gaps" if $a_n = b_n = 0$ except for $n \in \{n_k\}$ where $n_{k+1} - n_k \geq q > 1$. This subject is treated briefly in [40, chap. V, §9] and [1, chap.XI, §13].

Let f be a real function defined on T. We have defined ΛBV in §1. We now define the classes φBV and $V[h]$.

$\{I_k\}$ once again will denote any collection of nonoverlapping intervals in T. Let $\varphi(x)$ be a non-negative convex function defined on $[0, \infty]$ such that $\varphi(x)/x \to 0$ as $x \to 0$. We say that f is of φ-bounded variation (φBV) if for some $c > 0$, $\sup \{\sum \varphi(c|f(I_k)|) \mid \{I_k\}\} = V_c(f) < \infty$.

If $h(n)$ is a positive nondecreasing concave downward function on the positive integers, we say that $f \in V[h]$ if there is a constant C such that

$\sum_1^n |f(I_k)| \leq Ch(n)$, $n = 1,2,\ldots$, for every collection $\{I_k\}$.

We suppose that f is a real function in $L^1(T)$ with Fourier series

$$\Sigma\, c_{n_k} e^{in_k x}\ , \quad n_{-k} = -n_k,$$

satisfying $n_{k+1}-n_k \geq q > 1$, $k = 1,2,\ldots$. Let $I \subset T$ be a closed interval with length .

$$|I| = (1 + \delta)\ 2\pi/q\ , \quad \delta > 0.$$

With P. Isaza we have obtained the following results [4].

Theorem 1. With f and I as above,

(i) $f \in V[h]$ on I implies $c_n = O(h(|n|)/n)$.

(ii) $f \in \Lambda BV$ on I implies $c_n = O(1/\sum_{1}^{|n|} 1/\lambda_i)$.

(iii) $f \in \varphi BV$ on I implies $c_n = O(\varphi^{-1}(1/|n|))$.

Theorem 2. Let f and I be as above. Let $\omega_I(f,t)$ be the modulus of continuity of f restricted to I. If $f \in V[n^\alpha]$ on I, $0 \leq \alpha < \frac{1}{2}$ and

$$\sum_{n=1}^{\infty} \frac{1}{n}\, \omega_I^{\frac{1-2\alpha}{2(1-\alpha)}}(f,\tfrac{1}{n}) < \infty,$$

then the Fourier series of f converges absolutely.

It is clear that if we make the assumption $n_{k+1} - n_k \to \infty$, then the conclusion holds for any nondegenerate interval I.

3. CONJUGATE FUNCTIONS AND THE BOHR-PÁL THEOREM

Let us suppose that f is a real, continuous function on T. Although S(f), the Fourier series of f, may diverge on a nonempty set of measure zero, the Bohr-Pál theorem asserts that, after a suitable change of variable has been made, the Fourier series of the new function will converge uniformly. In other words, there is a strictly increasing, continuous function g, mapping $[-\pi, \pi]$ onto itself, so that $S[f\circ g]$ converges uniformly.

The classical proof of this result is due to Salem [6, 10 vol. I, p. 294] and uses complex methods. If f had the property that for only one $a \in (-\pi, \pi)$ is $f(-\pi) = f(a) = f(\pi)$ and $f(x) - f(a)$ has opposite signs in the intervals $(-\pi, a)$ and (a, π), then a simple closed curve C may be defined by $f(t) + i\varphi(t)$ for a suitable function φ and, by using a conformal map of $|z| < 1$ onto the interior of C, we find that we can also give C by an equation $w = F(e^{i\theta})$ where $F(z)$ is regular on $|z| < 1$, continuous on $|z| = 1$, and $S[F(e^{i\theta})]$ converges uniformly. The Bohr-Pál theorem is obtained by expressing the given function as the sum of a continuous function of bounded variation and a function of the type just described.

With W. Jurkat [5] we have obtained the following improvement of the Bohr-Pál theorem.

THEOREM. If f is a real, continuous function on T, there is a strictly increasing, continuous function g, mapping $[-\pi, \pi]$ onto itself, so that the conjugate of $f\circ g$ is continuous and of bounded variation on $[-\pi, \pi]$.

An immediate consequence of this is:

COROLLARY. If f is a real, continuous function on T, there is a strictly increasing, continuous function g, mapping $[-\pi, \pi]$ onto itself, so that if

$$S[f\circ g] = \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx + b_k \sin kx),$$

then, as $n \to \infty$,

(i) $n(|a_n| + |b_n|) = O(1)$

(ii) $\frac{1}{n} \sum_{k=1}^{n} k(a_k^2 + b_k^2)^{1/2} = o(1)$.

Given the theorem, applying the usual estimate of the Fourier coefficients of a function of bounded variation to the conjugate $f \circ g$ yields (i). Estimate (ii) is Wiener's necessary and sufficient condition for a function of bounded variation to be continuous applied to the conjugate $f \circ g$ [1 vol. I, p. 212; 10 vol. I, p. 108]. Note that the continuity of $f \circ g$ implies that $S[f \circ g]$ is uniformly (C, 1)-summable and, by (ii), also uniformly convergent, which is the Bohr-Pál theorem.

REMARK. From the fact that the conjugate of $h = f \circ g$ is continuous and of bounded variation one can deduce that $\tilde{h}$ and h belong to the Lipschitz classes $\lambda^p_{1/p}$ for $1 < p < \infty$ and also to Λ^1_* (see [10, vol. I, p. 45] for the definitions of these classes).

The proof of our theorem is based on the following result which is an analogous statement for R.

LEMMA. If f is a real, continuous function on R with bounded support, there is a strictly increasing continuous function g, mapping R onto R, such that the Hilbert transform of $f \circ g$,

(1) $$H[f \circ g](x) = \frac{1}{\pi} \text{P.V.} \int_R \frac{f \circ g(t)}{x-t}\, dt , \quad x \in R,$$

is continuous and of bounded variation on R.

Our proof of this result, like Salem's proof of the Bohr-Pal theorem, relies on conformal mapping.

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