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Higher Order Riemann Complete Integrals

1. Introduction. It is easy to show that the generalized Riemann integral [5], [7] integrates an everywhere finite ordinary derivative. Necessary and sufficient conditions for an exact Peano derivative to be generalized Riemann integrable were obtained in [4]. It is the purpose of this article to introduce the idea of a "higher order" generalized Riemann integral that will integrate a wide class of finite generalized derivatives.

2. Preliminaries. Let

$$a = x_0 < x_1 < \dots < x_m = b$$

be a division (partition) of the interval $[a,b]$, and suppose the numbers z_j are associated with the division by the relation $x_{j-1} \le z_j \le x_j$. Such a division, denoted by $\mathcal{D}$, is called a tagged division with tags z_j, $j = 1,2,\dots,m$. Suppose further there is given a function $\delta(x)$ such that $\delta(x) > 0$, $x \in [a,b]$. If the tagged division $\mathcal{D}$ has the property that $[x_{j-1},x_j] \subset (z_j-\delta(z_j),\ z_j+\delta(z_j))$, $j = 1,2,\dots,m$, then the division is said to be compatible with $\delta(x)$.

It has been shown by Henstock ([7] and [8]) that if $\delta(x) > 0$ is an arbitrary function defined on $[a,b]$, then there is a tagged division of $[a,b]$ compatible with $\delta(x)$.

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Moreover, given $\delta(x) > 0$ defined on $[a,b]$, there is a tagged division of $[a,b]$ compatible with $\delta(x)$ so that $x_0 = a$ is the tag of $[x_0,x_1]$ and $x_m = b$ is the tag for $[x_{m-1},x_m]$. (Cf [10], page 11 ff.)

If $\mathcal{D}$ is a tagged division of $[a,b]$ where, as above, the tags satisfy the relations

$$z_1 = a, \quad x_{k-1} \leq z_k \leq x_k, \quad 2 \leq k \leq m-1, \quad z_m = x_m,$$

we can obtain a finer tagged division of $[a,b]$ by dividing each interval $[x_{j-1},x_j]$ into subintervals $[x_{j-1},z_j]$ and $[z_j,x_j]$ and taking z_j to be the tag of each of these subintervals for $j = 2,3,\ldots,n-1$. We incorporate the above considerations into a formal definition.

<u>Definition 2.1</u>. <u>A tagged division of</u> $[a,b]$ <u>will be called a restricted tagged division of</u> $[a,b]$ <u>if it has the form</u>

$$x_0 = z_1 < x_1 < z_2 < x_2 < z_3 < x_3 < \ldots < x_{m-2} < z_{m-1} < x_{m-1} < z_m = x_m$$

<u>where</u> $x_0 = z_1$ <u>is the tag of</u> $[x_0,x_1]$, $x_m = z_m$ <u>is the tag of</u> $[x_{m-1},x_m]$ <u>and</u> z_j <u>is the tag of both</u> $[x_{j-1},z_j]$ <u>and</u> $[z_j,x_j]$ <u>for</u> $j = 2,3,\ldots,m-1$. <u>If a restricted tagged division of</u> $[a,b]$ <u>has further the property that</u> $z_j-x_{j-1} = x_j-z_j$, $j = 2,3,\ldots,m-1$, <u>the division will be called a restricted symmetric tagged division of</u> $[a,b]$.

It is clear that given $\delta(x) > 0$ defined on $[a,b]$ there exists a restricted tagged division of $[a,b]$ compatible with $\delta(x)$. That there exists a restricted <u>symmetric</u> tagged division of $[a,b]$ compatible with $\delta(x)$ follows from [9].

Definition 2.2. (cf. [7] and [10]) Let f be defined (and finite) on $[a,b]$. The number I is the definite generalized Riemann integral (or the Riemann complete integral) of f on $[a,b]$ if, corresponding to $\epsilon > 0$ there is a $\delta(x) > 0$, $x \in [a,b]$ so that

$$\left| I - \sum_{j=1}^{m} f(z_j)(x_j - x_{j-1}) \right| < \epsilon$$

for each tagged division $\mathcal{D}$ compatible with δ. When the integral exists we write $I = RC \int_a^b f(t)dt$.

Note: It is clear that we would obtain the same integral if we substituted restricted tagged divisions for tagged divisions in the definition. This is because $f(z_j)(x_j - x_{j-1}) = f(z_j)(x_j - z_j) + f(z_j)(z_j - x_{j-1})$. (cf. [7], page 84). This will not be true of the generalized integrals we define in this paper.

3. Generalized derivatives. Let $F(x)$ be a function defined on $[a,b]$. If there are constants $\beta_0, \beta_2, \ldots, \beta_{2r}$, depending on x_0 but not on h, such that

$$\frac{1}{2}\{F(x_0+h)+F(x_0-h)\} - \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} = o(h^{2r}), \tag{3.1}$$

as $h \to 0$, then β_{2r} is called the de la Vallée Poussin (or generalized symmetric) derivative of order $2r$ of $F(x)$ at $x = x_0$, and is written $D^{2r}F(x_0)$. If $D^{2r}F(x_0)$ exists then $D^{2k}F(x_0)$, $0 \le k \le r-1$, exists and $\beta_{2k} = D^{2k}F(x_0)$, $0 \le k \le r$.

Odd numbered symmetric derivatives are defined in a similar way:

If there are constants $\beta_1, \beta_3, \ldots, \beta_{2r+1}$, depending on x_0 but not on h, such that

$$\frac{1}{2}\{F(x_0+h)-F(x_0-h)\} - \sum_{k=0}^{r} \beta_{2k+1} \frac{h^{2k+1}}{(2k+1)!} = o(h^{2r+1}), \tag{3.2}$$

as $h \to 0$, then β_{2r+1} is the de la Vallée Poussin derivative of order $(2r+1)$ of $F(x)$ at $x = x_0$, and is written $D^{2r+1}F(x_0)$.

Now let $F(x)$ be a continuous function defined on $[a,b]$. If there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that

$$F(x_0+h)-F(x_0)-h\alpha_1-\ldots-\frac{h^n}{n!}\alpha_n = o(h^n), \tag{3.3}$$

as $h \to 0$, then α_n is called the n^{th} Peano derivative of F at x_0 and is denoted by $F_n(x_0)$ ([6] and [11]). If F has an n^{th} Peano derivative at x_0, then F has also a k^{th} Peano derivative at x_0, $F_k(x_0)$, $k = 1,2,\ldots,n-1$, where $F_1(x_0) = F'(x_0)$. It is clear that the existence of $F_{(r)}(x_0)$ implies that of $D^rF(x_0)$ and that $D^rF(x_0) = F_r(x_0)$.

If $F_k(x_0)$ exists for $1 \le k \le n-1$, define $\gamma_n(F;x_0,h)$ by

$$\gamma_n(F;x_0,h) = \frac{n!}{h^n}\left\{F(x_0+h)-F(x_0) - \sum_{k=1}^{n-1}\frac{h^k}{k!}F_k(x_0)\right\}. \tag{3.4}$$

If $h\gamma_n(F;x_0,h) \to 0$, as $h \to 0$, then F will be said to be P_n-continuous at x_0.

J.C. Burkill [2] constructed a scale of (Cesàro-Perron) integrals, the C_kP-integrals, $k = 0,1,2,\ldots,n,\ldots,$ in which the C_0P-integral is the

Perron integral which, it is known, is equivalent to the Riemann complete integral.

Suppose that for $0 \le k \le n-1$ the C_kP-integral has been defined. A function $M(x)$, defined on $[a,b]$ is said to be C_n-continuous on $[a,b]$ if it is $C_{n-1}P$-integrable over $[a,b]$ and if

$$C_n(M,x,x+h) \equiv \left(\frac{n}{h^n}\right) C_{n-1}P \int_x^{x+h} (x+h-t)^{n-1} M(t)\,dt \to M(x),$$

as $h \to 0$, for every x in $[a,b]$. Let

$$C_n\bar{D}M(x) \equiv \overline{\lim_{h\to 0}} \left(\frac{C_n(M,x,x+h)-M(x)}{h/n+1}\right)$$

and define $C_n\underline{D}M(x)$ in the obvious way. If $C_n\bar{D}M(x) = C_n\underline{D}M(x)$ then the common value is taken to be the C_n-derivative, $C_nDM(x)$.

The functions $M(x)$ and $m(x)$ are called C_nP-major and minor functions, respectively, if $f(x)$ over $[a,b]$ if

(3.5) $M(x)$ and $m(x)$ are C_n-continuous in $[a,b]$;

(3.6) $M(a) = m(a) = 0$;

(3.7) $C_n\underline{D}M(x) \ge f(x) \ge C_n\bar{D}m(x), \quad x \in [a,b]$;

(3.8) $C_n\underline{D}M(x) \ne -\infty, \quad C_n\bar{D}m(x) \ne +\infty$.

If for every $\epsilon > 0$, there is a pair $M(x)$, $m(x)$ satisfying the conditions (3.5), (3.6), (3.7) and (3.8) above and such that $|M(b)-m(b)| < \epsilon$, then $f(x)$ is said to be C_nP-integrable in $[a,b]$ and

$$C_nP\int_a^b f(t)dt = \inf M(b) = \sup m(b),$$

where the inf and sup are taken over all major and minor functions, respectively.

It follows easily that the C_nP-integral integrates an everywhere finite C_n-derivative, and

$$g(b)-g(a) = C_nP\int_a^b C_nDg(x)dx.$$

Moreover, if $f(x)$ is $C_{n-1}P$-integrable on $[a,b]$, then $f(x)$ is C_nP-integrable on $[a,b]$, and the integrals agree. In particular, if f is RC-integrable then it is C_kP-integrable for $k = 1,2,3,\ldots,$ and the integrals all have the same value.

If f is $C_{r-1}P$-integrable on $[a,b]$, and if $r \geq 1$, the symmetric Cesaro derivative, or, SC_r-derivative, of f at x_0, $x_0 \in (a,b)$, denoted by $SC_rDf(x_0)$ is defined to be

$$\lim_{h\to 0}\left(\frac{C_r(f;x_0,x_0+h)-C_r(f;x_0,x_0-h)}{2h/r+1}\right)$$

if the limit exists (cf. [1] and [3]).

If f has an exact Peano derivative f_n for $x \in [a,b]$, we have at each x, corresponding to $\epsilon > 0$, a $\delta(x,\epsilon) > 0$ so that

$$\left|f(t)-f(x) - \sum_{k=1}^{n-1}\frac{f_k(x)(t-x)^k}{k!} - \frac{(t-x)^n}{n!}f_n(x)\right|$$

$$\leq \begin{cases}\theta(t,x), & x-\delta(x,\epsilon) < t < x\\ \theta(x,t), & x < t < x+\delta(x,\epsilon)\end{cases},$$

where $\theta(u,v) = \frac{\epsilon(v-u)^n}{n!}$, $a \le u \le v \le b$, is a non-negative and finitely superadditive interval function with $\theta(a,b)$ small for small $\epsilon > 0$, since for positive integer n,

$$(c-a)^n = [(c-b)+(b-a)]^n = (c-b)^n+(b-a)^n + \text{positive terms}$$
$$\ge (c-b)^n+(b-a)^n, \quad a \le b \le c.$$

Assuming $f_k(x)$ exists, $0 \le k \le n-1$, we could define an integral as follows:

If there is a number I such that for all $\epsilon > 0$ there exists a function $\delta(x) \equiv \delta(x,\epsilon) > 0$ for $x \in [a,b]$ such that

$$\left| I - \sum_{k=1}^{n-1} \sum_{j=1}^{m} \frac{f_k(z_j)}{k!} (x_j - x_{j-1})^k - \sum_{j=1}^{m} \frac{(x_j-x_{j-1})^n}{n!} g(z_j) \right| < \epsilon$$

for all tagged divisions compatible with $\delta(x)$, then I is the integral of g over $[a,b]$.

Then if $f_n(x)$ exists in $[a,b]$ it follows that corresponding to $\epsilon > 0$ there is a function $\delta(x) > 0$, $x \in [a,b]$ such that

$$\left| f(b)-f(a)- \sum_{k=1}^{n-1} \sum_{j=1}^{m} \frac{f_k(x_{j-1})}{k!} (x_j-x_{j-1})^k - \sum_{j=1}^{m} \frac{(x_j-x_{j-1})^n}{n!} f_n(x_{j-1}) \right|$$

$$= \left| \sum_{j=1}^{m} \left\{ f(x_j)-f(x_{j-1}) - \sum_{k=1}^{n-1} \frac{f_k(x_{j-1})(x_j-x_{j-1})^k}{k!} - \frac{f_n(x_{j-1})(x_j-x_{j-1})^n}{n!} \right\} \right|$$

$$\le \frac{\epsilon}{n!} \sum_{j=1}^{m} (x_j-x_{j-1})^n \le \frac{\epsilon(b-a)^n}{n!},$$

for all divisions compatible with $\delta(x)$, and so

$$f(b)-f(a) = \int_a^b f_n(t)\,dt.$$

In fact this reduces to simply the generalized Riemann integral of $f_1(x)$ since all the sums $\sum_{j=1}^{m} \frac{f_k(x_{j-1})}{k!}(x_j-x_{j-1})^k$, $k = 2,3,\ldots,n-1$ and $\sum_{j=1}^{m} \frac{f_n(x_{j-1})}{n!}(x_j-x_{j-1})^n$ have limit 0, as we shall show in Theorem 3.1 to follow. Thus nothing is gained by introducing this style of integration and so we take another approach in section 4.

Theorem 3.1. Let $f(x)$ be defined on $[a,b]$ and suppose $p > 0$. Then given $\epsilon > 0 \ \exists \ \delta(x) > 0$ such that

$$\left|\sum_{j=1}^{m} f(z_j)(x_j-x_{j-1})^{1+p}\right| < \epsilon,$$

for all tagged divisions compatible with δ.

Proof. It is easy to verify that we may choose $\delta(x) = \epsilon^{1/p}\{(b-a)(1+|f(x)|)\}^{-1/p}$.

4. Higher order Riemann complete integrals. If f is a finite function defined on $[a,b]$, let two interval functions be defined by $F_\ell(u,v) \equiv F_\ell(f,u,v) \equiv f(v)(v-u)$ and $F_r(u,v) \equiv F_r(f,u,v) \equiv f(u)(v-u)$. It will be convenient to denote a pair of interval functions by a single letter in script face. For example we shall write $\mathcal{F}(u,v) = \{F_\ell(u,v), F_r(u,v)\}$ or, more briefly, $\mathcal{F} = (F_\ell, F_r)$. In the following we shall introduce a variety of interval function all of which, like the above, are defined in terms of a given point function f and its derivatives.

Given a restricted tagged division $\mathcal{D}$ of $[a,b]$ and a pair of interval functions (φ_ℓ,φ_r) we shall consider sums of the form

$$\varphi_r(z_1,x_1)+\varphi_\ell(x_1,z_2)+\varphi_r(z_2,x_2)+\varphi_\ell(x_2,z_3)+\ldots+\varphi_r(z_{m-1},x_{m-1})+\varphi_\ell(x_{m-1},z_m).$$

We shall denote such sums by $(\mathcal{D})\Sigma\varphi_\sigma$ where $\sigma = \ell$ or r depending on whether the tag of the interval is the right hand or left hand end point.

We introduce below the definitions of the generalized Riemann complete integral and the generalized symmetric Riemann complete integral.

Definition 4.1. The number I will be called the generalized Riemann complete (generalized symmetric Riemann complete) integral of f with respect to the pair of interval functions $h(u,v) = \{h_\ell(u,v),h_r(u,v)\}$ on $[a,b]$ if, corresponding to $\epsilon > 0$, there is a function $\delta(x) > 0$ so that

$$(4.1)\qquad |I-(\mathcal{D})\Sigma(h_\sigma+F_\sigma)| < \epsilon$$

for all restricted tagged divisions (restricted symmetric tagged divisions), $\mathcal{D}$ compatible with $\delta(x)$, where $\sigma = \ell$ or r, depending on whether the tag of the interval is the right hand or left hand end point.

The notation for these integrals is

$$I = (\mathrm{GRC},h)\int_a^b f(t)\,dt \qquad \text{and} \qquad I = (\mathrm{GSRC},h)\int_a^b f(t)\,dt,$$

respectively.

Theorem 4.1. $f(x)$ is Riemann complete integrable if and only if it is generalized Riemann complete integrable with respect to $h \equiv \{0,0\}$ and the integrals are equal.

Proof. If $h_\ell = h_r = 0$, the sums (4.1) are the defining sums for the Riemann complete integral.

Corollary. If $F'(x) = f(x)$ where $f(x)$ is finite for all x, then

$$(GRC,h)\int_a^b f(t)\,dt = RC\int_a^b f(t)\,dt = F(b)-F(a), \quad \text{with } h \equiv \{0,0\}.$$

It is clear also that if f is generalized Riemann complete integrable with respect to $h = \{h_h, h_r\}$ then f is generalized symmetric Riemann complete integrable with respect to h and the integrals are equal.

5. Integration of the Peano and C_n-derivatives. First we assume that $f_n(x)$ and $C_nDf(x)$ exist in $[a,b]$ and define several particular interval functions:

$$G_\ell(f,n,u,v) = \left\{-(n-1)f_{n-1}(v)-f_{n-1}(u)-n!\left[\frac{f(u)-\sum_{k=0}^{n-2}\frac{(u-v)^k}{k!}f_k(v)}{(-1)^n(v-u)^{n-1}}\right]\right\},$$

$$G_r(f,n,u,v) = \left\{(n-1)f_{n-1}(u)+f_{n-1}(v)-n!\left[\frac{f(v)-\sum_{k=0}^{n-2}\frac{(v-u)^k}{k!}f_k(u)}{(v-u)^{n-1}}\right]\right\},$$

$$F_\ell(f,n,u,v) = f_n(v)(v-u),$$

$$F_r(f,n,u,v) = f_n(u)(v-u),$$

$$H_\ell(f,n,u,v) = (n+1)C_n(f,v,u)-nf(v)-f(u)$$

$$H_r(f,n,u,v) = -(n+1)C_n(f,u,v)+nf(u)+f(v).$$

Theorem 5.1. *If* $f(x)$ *has a finite Peano derivative* $f_n(x)$ *in* $[a,b]$ *then* $f_n(x)$ *is generalized Riemann complete integrable with respect to* $\mathcal{G} = (G_\ell, G_r)$ *on* $[a,b]$ *and*

$$f_{n-1}(b)-f_{n-1}(a) = (GRC,\mathcal{G}) = \int_a^b f_n(t)\,dt.$$

Proof. Given $\epsilon > 0$ there exists $\delta(x) > 0$ such that

$$|f_{n-1}(x)-f_{n-1}(t)-G_\ell(f,n,t,x)-F_\ell(f,n,t,x)|$$

$$= \left|\frac{n!}{(-1)^n(x-t)^{n-1}}\left\{f(t)-\sum_{k=0}^{n-1}\frac{f_k(x)(t-x)^k}{k!}\right\} - f_n(x)(x-t)\right| < \epsilon(x-t),$$

if $x-\delta(x) \le t < x$, since $f_n(x)$ exists.

In a similar way it can be shown that

$$|f_{n-1}(u)-f_{n-1}(x)-G_r(f,n,x,u)-F_r(f,n,x,u)| < \epsilon(u-x),$$

if $x < u \le x+\delta(x)$.

As we noted in Section 2, there exists a restricted tagged division compatible with $\delta(x)$. If $\mathcal{D}$ is one such division, say, $a = x_0 < x_1 <\ldots< x_{m-1} = x_m = b$, we can write (where $\sigma = r$ or ℓ

depending on whether the tag is the left hand or right hand end point of the corresponding interval):

$$|f_{n-1}(b)-f_{n-1}(a)-(\mathcal{D})\sum_{j=1}^{m}\{G_\sigma(f,n,x_{j-1},x_j)+F_\sigma(f,n,x_{j-1},x_j)\}|$$

$$\leq (\mathcal{D})\sum_{j=1}^{m}|f_{n-1}(x_j)-f_{n-1}(x_{j-1})\{-G_\sigma(f,n,x_{j-1},x_j)-F_\sigma(f,n,x_{j-1},x_j)\}|$$

$$< \epsilon\sum_{j=1}^{m}(x_j-x_{j-1}) = \epsilon(b-a).$$

This shows that $I = f_{n-1}(b)-f_{n-1}(a)$ is the generalized Riemann complete integral of $f_n(x)$ with respect to $\mathcal{G}$.

Theorem 5.2. If $f(x)$ has a finite C_nD-derivative on $[a,b]$ then $C_nDf(x)$ is generalized Riemann complete integrable with respect to $\mathcal{H} = \{H_\ell, H_r\}$ on $[a,b]$ and

$$f(b)-f(a) = (GRC,\mathcal{H})\int_a^b C_nDf(t)dt.$$

Proof. The method of proof is the same as for Theorem 5.1.

6. Integration of the de la Vallée Poussin and the SC_n-derivatives. In addition to the interval functions G_ℓ, G_r, H_ℓ, H_r introduced in the previous section we need the following interval functions (where we assume that $D^nf(x)$ and $SC_nDf(x)$ are defined in (a,b) and $f_{n-1}(x)$ in $[a,b]$):

$$J_r(f,2n,u,v) = (2n-1)f_{2n-1}(u)+f_{2n-1}(v)-(2n)!\left\{\frac{f(v)-\sum_{r=0}^{2n-2}\frac{f_r(u)(v-u)^r}{r!}}{(v-u)^{2n-1}}\right\},$$

$$J_\ell(f,2n,u,v) = -(2n-1)f_{2n-1}(v)-f_{2n-1}(u)-(2n)!\left\{\frac{f(u)-\sum_{r=0}^{2n-2}\frac{f_r(v)(u-v)^r}{r!}}{(v-u)^{2n-1}}\right\},$$

$$K_r(f,n,u,v) = -(n+1)C_n(f,u,v)+nF(u)+F(v),$$

$$K_\ell(f,n,u,v) = (n+1)C_n(f,v,u)-nF(v)-F(u),$$

for $a \le u < v \le b$,

$$I_r(f,2n,u,v) = D^{2n}f(u)(v-u)$$

$$I_\ell(f,2n,u,v) = D^{2n}f(v)(v-u),$$

for $a < u < v < b$, and

$$I_r(f,2n,a,v) = I_\ell(f,2n,u,b) = 0.$$

Theorem 6.1. Suppose $f(x)$ has a finite de la Vallée Poussin derivative of order $2n$ in (a,b), a finite Peano derivative of order $2n-1$ in $[a,b]$ and is P_{2n}-continuous at a and b. Then the function $g(x)$ defined by

$$g(x) = \begin{Bmatrix} D^{2n}f(x), & x \in (a,b) \\ 0, & x = a,b \end{Bmatrix},$$

is generalized symmetric Riemann complete integrable with respect to the interval functions $\mathcal{J} = (J_\ell, J_r)$ on $[a,b]$, and

$$(6.1) \qquad f_{2n-1}(b) - f_{2n-1}(a) = (\text{GSRC}, \mathcal{J}) \int_a^b g(x)\,dx.$$

Proof. Since $D^{2n}f(x)$ exists in (a,b) and $f(x)$ is P_{2n}-continuous at a and b, we have that corresponding to $\epsilon > 0$ there exists $\delta(x) \equiv \delta(x,\epsilon) > 0$ such that

$$(6.2) \qquad \left| \left\{ \frac{f(x+h) + f(x-h) - 2\sum_{r=0}^{n-1} \frac{D^{2r}f(x)h^{2r}}{(2r)!}}{h^{2n-1}/(2n)!} \right\} - (2D^{2n}f(x))(h) \right| < \epsilon h,$$

for $a < x-h < x < x+h < b$ and $h < \delta(x)$,

$$(6.3) \qquad \left| \frac{f(a+h) - f(a) - \sum_{k=1}^{2n-1} \frac{f_k(a)h^k}{k!}}{h^{2n-1}/(2n)!} \right| < \epsilon,$$

for $a < a+h < a+\delta(a) < b$, and

$$(6.4) \qquad \left| \frac{f(b-h) - f(b) - \sum_{k=1}^{2n-1} \frac{f_k(b)(-h)^k}{k!}}{h^{2n-1}/(2n)!} \right| < \epsilon,$$

for $a < b-\delta(b) < b-h < b$.

We have seen in Section 2 that there is a symmetric tagged division of $[a,b]$ which is compatible with $\delta(x)$. If $\mathcal{D}$ is one such division, say,

$$a = x_0 < x_1 <\ldots< x_{m-1} < x_m = b,$$

(where by the definition of a symmetric tagged division, m is even) then a is the tag for the interval $[a,x_1]$ and $x_1 < a+\delta(a)$, b is the tag for the interval $[x_{m-1},b]$ and $x_{m-1} > b-\delta(b)$, and x_2 is the tag of the two intervals $[x_1,x_2]$ and $[x_2,x_3]$ where $x_3-x_2 = x_2-x_1$, and so on.

Consequently we may write (where $\sigma = r$ or ℓ depending on whether the tag is the left hand or right hand end point of the corresponding interval);

$$\left|f_{2n-1}(b)-f_{2n-1}(a)-(\mathcal{D})\sum_{j=1}^{m}\{J_\sigma(f,2n,x_{j-1},x_j)+I_\sigma(2n,x_{j-1},x_j)\}\right|$$

(6.5)

$$\leq \left|(\mathcal{D})\sum_{j=1}^{m}\{f_{2n-1}(x_j)-f_{2n-1}(x_{j-1})-J_\sigma(f,2n,x_{j-1},x_j)-I_\sigma(2n,x_{j-1},x_j\}\right|.$$

We then have

$$\left|f_{2n-1}(x_1)-f_{2n-1}(a)-(2n-1)f_{2n-1}(a)-f_{2n-1}(x_1)+(2n)!\left(\frac{f(x_1)-\sum_{r=0}^{2n-2}\frac{f_r(a)(x_1-a)^r}{r!}}{(x_1-a)^{2n-1}}\right)\right|$$

$$= \left|(2n)!\frac{f(x_1)-\sum_{r=0}^{2n-1}\frac{f_r(a)(x_1-a)^r}{r!}}{(x_1-a)^{2n-1}}\right| < \epsilon \quad ,$$

by (6.3), and similarly, by (6.4), for the last term of the series.

Taking the remaining terms in pairs on the intervals $[x_{k-1},x_k]$ and $[x_k,x_{k+1}]$, $k = 2,4,\ldots,m-2$, with x_k as the tag in each case we see that

$$\sum_k \Big| f_{2n-1}(x_k)-f_{2n-1}(x_{k-1})-J_\ell(f,2n,x_{k-1},x_k)-I_\ell(2n,x_{k-1},x_k)$$

$$+f_{2n-1}(x_{k+1})-f_{2n-1}(x_k)-J_r(f,2n,x_k,x_{k+1})-I_r(2n,x_k,x_{k+1}) \Big|$$

$$= \sum_k \left| (2n)! \left(\frac{f(x_{k-1}) - \sum\limits_{r=0}^{2n-2} \dfrac{f_r(x_k)(x_{k-1}-x_k)^r}{r!}}{(x_k-x_{k-1})^{2n-1}} \right) -D^{2n}f(x_k)(x_k-x_{k-1}) \right.$$

$$\left. +(2n)! \left(\frac{f(x_{k+1}) - \sum\limits_{r=0}^{2n-2} \dfrac{f_r(x_k)(x_{k+1}-x_k)^r}{r!}}{(x_{k+1}-x_k)^{2n-1}} \right) -D^{2n}f(x_k)(x_{k+1}-x_k) \right|$$

$$= \sum_k \left| (2n)! \left(\frac{f(x_{k-1})+f(x_{k+1})-2\sum\limits_{r=0}^{n-1} \dfrac{D^{2r}f(x_k)(x_{k+1}-x_k)^{2r}}{(2r)!}}{(x_{k+1}-x_k)^{2n-1}} \right) -2D^{2n}f(x_k)(x_{k+1}-x_k) \right|$$

$$< \sum_k \epsilon\, 2(x_{k+1}-x_k),$$

by (6.2), since $x_{k+1}-x_k = x_k-x_{k-1}$ and $D^{2k}f(x) = f_{2k}(x)$.

The full sum in (6.5) is thus less than $\{\epsilon(b-a)+2\}$.

It is clear that the same kind of result holds for odd numbered symmetric derivatives $D^{2n+1}f(x)$, where,

$$J_r(f,2n+1,u,v) = (2n)f_{2n}(u)+f_{2n}(v)-(2n+1)!\left(\frac{f(v)-\sum_{r=1}^{2n-1}\frac{f_r(u)(v-u)^r}{r!}}{(v-u)^{2n}}\right),$$

and,

$$J_\ell(f,2n+1,u,v) = (-2n)f_{2n}(v)-f_{2n}(u)+2n+1)!\left(\frac{f(u)-\sum_{r=1}^{2n-1}\frac{f_r(v)(u-v)^r}{r!}}{(v-u)^{2n}}\right).$$

Theorem 6.2. Suppose $f(x)$ *has a finite* SC_n*-derivative in* (a,b) *and is* C_n*-continuous at* a *and* b*. Then the function* $h(x)$ *defined by*

$$h(x) = \begin{cases} SC_nDf(x), & x \in (a,b) \\ 0, & x = a,b \end{cases}$$

is generalized symmetric Riemann complete integrable with respect to the interval functions $\mathcal{K} = \{K_\ell, K_r\}$ *on* $[a,b]$ *and*

$$f(b)-f(a) = (GSRC,\mathcal{K})\int_a^b h(x)dx.$$

Proof. The method of proof is the same as for Theorem 6.1.

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