TOPICAL SURVEY *Real Analysis Exchange Vol. 11 (1985-86)*

Z. Piotrowski
Department of Mathematics & Computer Sciences
Youngstown State University
Youngstown, Ohio 44555, USA

Separate and Joint Continuity [1)]

0. Introduction. To orient the reader to the major concepts and present a few open questions let us start with the following three general problems.

Let X and Y be "nice" [2)] topological spaces, let M be metric and let $f : X \times Y \to M$ be separately continuous, that is, f is continuous with respect to each variable while the other is fixed.

I. Existence Problem: Find the set C(f) of points of continuity of f. If X and Y are "nice", then C(f) is usually a dense G_δ subset of $X \times Y$.

There is also interest in a

"Fiber version". It is the same as above, except now we look for C(f) in $\{x\} \times Y$, for any fixed x in X.

---

1) Originally presented as an invited address during IX Summer Symposium on Real Analysis, June 12 - 15, 1985, Louisville, KY.
2) For example, *Polish spaces* (=separable complete metric)

II. Characterization Problem: Characterize C(f) as a subset of $X \times Y$. Again for "nice" spaces X and Y, the set C(f) is usually the complement of an F_σ set contained in the product of two sets of first category.

III. Uniformization Problem: Find a "uniform", "thick" subset A of X such that $A \times Y$ is contained in C(f). Again, if X and Y are "nice" then A is usually a dense G_δ subset of X. The Uniformization Problem is also known as a *Namioka-type problem.* (See [Na].)

1. PREHISTORY. Leaving to historians of mathematics the job of determining who was *the first* to construct a separately continuous function $f: \mathbb{R}^2 \to \mathbb{R}$ which is not continuous at some fixed point, let us mention only that the earliest published example known to the author appeared in 1873, 26 years before Baire's [Ba]. Its author, J. Thomae [Th], wrote

> ... "Dann müsste z. B. die Function $\omega(y, z) = \sin 4 \operatorname{arc\,tg} \frac{y}{z}$, welche wir für $z = 0$ dadurch definiren, dass wir sie längs der ganzen y-Achse (in der y, z-Ebene) gleich Null annehmen, im Innern des Kreises $y^2 + z^2 = 1$ überall stetig sein." ...

which shows that he knew of the existence of a function continuous along every straight line through every point in its domain[1)] which is not continuous. He also states that these phenomena were known earlier to E. Heine (1815-1897). (See also [Pr] and [Rs].)

The 1884 Calculus textbook [Ge] (!) by A. Genocchi, *con aggiunte* with G. Peano, contains the now standard examples [Ru] of functions which are

---

1) This type of "almost continuity" (known also as "linear continuity") has been subsequently studied in [Lb], pp. 199-200, [Ko], [KV1], [KV2] and [Sl].
Let us also mention that in the sixties a similar class of functions $f: \mathbb{R}^2 \to \mathbb{R}$ (namely those that are continuous along *almost* all lines in every direction) was studied by W. H. Fleming, J. Serrin and D. Waterman. Finally C. Coffman [Co] characterized this class of functions in terms of their partial derivatives.

separately continuous or are continuous along all lines in every direction but are not continuous at the origin (0,0).

Due to an unprecedentedly careful way of quotation of new results in [Ge] we can be sure that these examples appear for the first time.

2. History – from R. Baire through H. Hahn

Given a function $f\colon \prod_{i=1}^{n} X_i \to Z$, we shall denote that f is *separately continuous* by $f\colon \prod_{i=1}^{n} X_i \overset{s}{\rightarrow} Z$.

Let us briefly recall the main results of R. Baire [Ba] concerning our topic:

(*) Given $f\colon [0,1] \times [0,1] \overset{s}{\rightarrow} \mathbb{R}$, then there is a residual set of lines parallel to each axis consisting entirely of continuity points.

(**) If $f\colon \mathbb{R} \times \mathbb{R} \overset{s}{\rightarrow} \mathbb{R}$, then for every point $(x_0,y_0) \in \mathbb{R} \times \mathbb{R}$, for every disc K centered at (x_0,y_0) and for every $\varepsilon > 0$, there is a disc K_1 contained in K such that $|f(x,y) - f(x_0,y_0)| < \varepsilon$ for every (x,y) from K_1.[1)]

$({}^{**}_{\;*})$ There are functions $f\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \overset{s}{\rightarrow} \mathbb{R}$ which are discontinuous at *every* point of certain lines.

$({}^{**}_{**})$ A function $f\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \overset{s}{\rightarrow} \mathbb{R}$ may be of the second class of Baire but no worse.

Somewhat similar topics, although involving for example partial derivatives, have been studied in [VV].

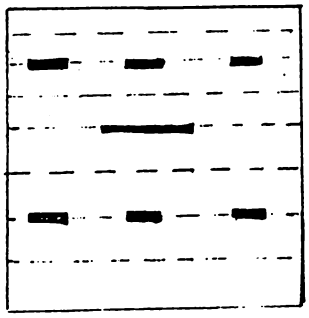
---

1) This observation is due to G. Volterra [Vo]. The property of separately continuous functions just presented was later called *quasi-continuity* [Kp]. See also [Mt], [Nb1], [Nb2] and [Pt1] for further generalizations.

An interesting process of densifying the set D(f) of points of discontinuity of separately continuous functions is shown by G. C. Young and W. H. Young [YY], namely:

There is a function $f : [0,1] \times [0,1] \to \mathbb{R}$ which is continuous with respect to every straight line and which has *uncountably* many points of discontinuity in *every* rectangle contained in the unit square.[1)]

Sketch of the construction:



Place the Cantor ternary set on the line $y = \frac{1}{2}$. On each of the lines $y = \frac{1}{4}$ and $y = \frac{3}{4}$ we place the Cantor ternary set with 3^2 as base, instead of 3. Generally, on all the lines $y = \frac{p}{q}$ of our set, where $q = 2^n$, we place Cantor sets with 3^n as base.

Let $f_n(x,y)$ be numerically less than 1, continuous with respect to every straight line and discontinuous *only* at the points of the (perfect and nowhere dense) set constructed on the n^{th} line.

Then

$$f(x,y) = \frac{1}{2} f_1(x,y) + \frac{1}{4} f_2(x,y) + \ldots + \frac{1}{2^n} f_n(x,y) + \ldots$$

is the required function. $\square$

Twenty years after the appearance of [Ba], H. Hahn [Hhl] improved some of Baire's results, namely:

(i) Given a function $f : \mathbb{R}^n \to \mathbb{R}$, then any (n-1)-dimensional hyperplane obtained by fixing one coordinate contains a dense set of continuity points of f. (Compare (*).)

1) In 1949 T. Tolstoff [To] showed that there is a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ whose set D(f) of points of discontinuity has a *positive Lebesgue measure*.

(ii) A function $f : \mathbb{R}^n \,\&\!\!\rightarrow \mathbb{R}$ is quasi-continuous[1)]; this is a natural extension of (**).

And thirdly :

(iii) A function $f : \mathbb{R}^n \,\&\!\!\rightarrow \mathbb{R}$ may be discontinuous at every point of some $(n-2)$-dimensional hyperplane. (Compare $(\underset{*}{**})$.)

In fact, let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be discontinuous at $(0,0)$. Then $f : \mathbb{R}^n \,\&\!\!\rightarrow \mathbb{R}$, where $f(x_1, x_2, \ldots, x_n) \equiv g(x_1, x_2)$, is discontinuous at every point of the $(n-2)$-dimensional hyperplane $x_1 = 0$, $x_2 = 0$.

The condition $(\underset{**}{**})$ of Baire has been strengthened by H. Lebesgue [Lb] to the following result:

(iv) A function $f : \mathbb{R}^n \,\&\!\!\rightarrow \mathbb{R}$ may be of class $n-1$ of Baire but no worse.

Some related studies of the distribution of points of continuity in hyperplanes are presented also in [Bg1] and [Bg2].

The famous text [Hh2] of H. Hahn is the first monograph, and the only so far, where the separate versus joint continuity problem receives so much attention. In fact, §39 (14 pages) is devoted completely to this topic.

Before we present some of his results let us make the following notational convention.

Given a function $f : \prod_{i=1}^{n} X_i \to Y$, we shall say that f is *weakly separately continuous*, denoted by $f : \prod_{i=1}^{n} \underline{X_i} \,\&\!\!\rightarrow Y$, if for all $x_i \in X_i$, $1 \leq i \leq n-1$ the sections f_{x_i} given by $f_{x_i}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_1, x_2, \ldots, x_n)$ are

---

1) $f : X \to Z$ is *quasi-continuous* if for every $x \in X$, for all open sets U and V containing respectively x and f(x), there is a nonempty open set U^1, $U^1 \subset U$, such that $f(U^1) \subset V$. There are quasi-continuous functions of arbitrary class of Baire [Mr].

continuous and for all $x_n \in D \subset \mathrm{Cl}\, D = X$ the sections f_{x_n} given by $f_{x_n}(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_2, \ldots, x_n)$ are continuous; Cl stands for the closure operator.

The following result of H. Hahn answers our Uniformization Problem.

Let $X_1, \ldots, X_{n-1}$ be metric Čech-complete spaces, X_n be compact metric and let $\prod_{i=1}^{n} X_i \looparrowright \mathbb{R}$. Then there is a residual set $A \subset \prod_{i=1}^{n-1} X_i$ such that $A \times X_n \subset C(f)$.

Further, [Hh2] offers systematic studies of so-called B-functions.[1)]

3. On the Existence Problem.

The following theorem due to F. Topsøe and J. Hoffman-Jørgensen [Rg] is based on an idea due to K. Kuratowski [Ku1].

Theorem 3.1 Let X be Hausdorff and let Y and M be metric. If $f\colon X \times Y \looparrowright M$ is a function, then $C(f)$ is a residual subset of $X \times Y$ such that all its y-sections $(= x \in X : (x,y) \in C(f),\ y \in Y)$ are residual in X.

The theorem given below has been proved independently by J. C. Breckenridge, T. Nishiura [BN] and myself [Pt2].

Theorem 3.2 Let X be Baire, Y be first countable and Z be metric. If $f : X \times Y \to Z$ has all its x-sections f_x continuous and all its y-sections f_y quasi-continuous, then $C(f)$ is a dense G_δ subset of $X \times \{y\}$, for any $y \in Y$.

The above result answers the "Fiber version" of the Existence Problem. It also generalizes [Bu] (where Y is *metric* and f is *separately continuous*). See also J. D. Weston's [We], where Y is first countable, Z is metric, $f : X \times Y \looparrowright Z$ and $C(f)$ is residual.

1) A somewhat similar notion known as *symmetric quasi-continuity* has been studied by S. Kempisty. See also [Mt], [Pt3] and [LeP2] for further generalizations.

Another result which ensures the existence of "many" points of continuity in $X \times Y$ can be derived from the following Baire-Lebesgue-Kuratowski-Montgomery [1)] theorem.

Theorem 3.3 Let X and Y be metric and let $f: X \times Y \to \mathbb{R}$ be continuous in x and of class α in y. Then f is of class $\alpha+1$.

In fact, if $\alpha = 0$, f is of class 1. Thus $C(f)$ is residual. Now, if $X \times Y$ is Baire, then $C(f)$ *is* a dense G_δ subset of $X \times Y$.

The following interesting result of W. Moran [Mo] is in the spirit of Theorem 3.3 and may ensure "many" points of continuity of separately continuous functions defined on the product of compact-like non-metrizable spaces. See [CaK] for further generalizations.

Theorem 3.4 A function $f: X \times Y \to \mathbb{R}$ from a product $X \times Y$ of compact spaces is the pointwise limit of a sequence of continuous functions on $X \times Y$ if and only if it is Baire measurable.

4. Cluster Sets and Continuity.

E. F. Collingwood [Co1], [Co2] observed that some of his results on the boundary behavior of functions meromorphic in the unit circle *do not* depend on the assumption that the considered objects are analytic functions, and these results can be carried over to more abstract spaces.

---

1) See [Ba], [Lb], [Ku2], [Ku4], [Mg]. Compare [Eg] where a short proof is given (using the fact that metric spaces have σ-locally finite bases).

Shortly thereafter J. D. Weston [We] presented an abstract theory of cluster sets. Let us follow his definition of the cluster set. Let T and Z be topological spaces. *The cluster set of a function* $f\colon T \to Z$ *at a point* $t \in T$, denoted $C(f;t)$, is defined as follows:

$$C(f;t) = \bigcap_{U \in \mathcal{U}_t} \operatorname{Cl} f(U), \text{ where } \mathcal{U}_t \text{ is the system of neighborhoods}$$

of t in T.

The following Lemma 4.1 is not hard to show.

Lemma 4.1 Let T be a topological space, let Z be compact and let $f\colon T \to Z$ be given. Then f is continuous if and only if for every $t \in T$ we have $C(f;t) = \{f(t)\}$.

With the help of the above Lemma he showed the following result. (See Section 3.)

Theorem 4.2 Let Y be first countable and let Z be compact metric. For every $f\colon X \times Y \to Z$ and for every $y \in Y$, the set $C(f)$ is residual in $X \times \{y\}$.

Feiock's result [Fk], being a careful analysis of Weston's proof, gives an answer to our Uniformization Problem.

Theorem 4.3 [Fk] Let Y be second countable and let Z be compact metric. If $f\colon X \times Y \to Z$, then there is a residual subset A of X such that $A \times Y \subset C(f)$.

Minor variations of the proof of Feiock were done by M.M. Mirzojan [Mz] where the following result is shown:

Theorem 4.4 Let Y be metric, locally compact and σ-compact and let Z be a compact metric space. For every function $f\colon X \times Y \to Z$ there is a residual G_δ subset A of X such that $A \times Y \subset C(f)$.

N. B. Mal'seva [M1] gives more examples of cluster sets of functions between topological spaces and provides an updated bibliography.

Before we present the next result let us recall that a space X is called <u>a k_ω-space</u> if $X = \bigcup_{n=1}^{\infty} X_n$ with X_n's being compact and increasing and X having their weak topology.
In fact Mirzojan's result has recently been generalized [LeP1] to one where Y is assumed to be a metric k_ω-space.

We shall now present some applications of the results on multifunctions to our general problem of separate versus joint continuity.

Let us start by formulating the following definition.

A function $f: X \to Y$ is called <u>*nearly continuous at* $x_o \in X$</u> if for every open set V containing $f(x_o)$, the point x_o is in the interior of the closure of $f^{-1}(V)$.

<u>Lemma 4.5</u>[1)] [Ke2] Let Y be second countable. Then *any* function $f: X \to Y$ is nearly continuous at every point of a residual subset of X.

<u>Theorem 4.6</u> [Ke2] Let Y be second countable and let Z be regular and second countable. If $f: X \times Y \underline{\leftrightarrow} Z$, then there is a residual set A in X such that $A \times Y \subset C(f)$.

Sketch of the proof: Let $\{U_i\}$ be a countable base for Y, let $\{V_j\}$ be a countable base for Z and let $\mathcal{A}$ be the countable system of sets $A_{ij} = \{h \in C(Y,Z) : h(U_i) \subset V_j\}$ i,j = 1, 2, 3,... where C(Y,Z) is the set of all continuous functions from Y to Z.

---

1) This result has been shown originally by H. Blumberg in 1922; see also [Pt4] and [Wi] for further generalizations.

Now let $g: X \to C(Y,Z)$ assign the function $f_{x_o} \in C(Y,Z)$ to each $x_o \in X$. By Lemma 4.5 there is a residual set A on which g is nearly continuous.

We shall show that the set A has the properties mentioned in the conclusion of Theorem 4.6.

In fact, take $(x_o,y_o) \in A \times Y$ and an open neighborhood V_j of $f(x_o,y_o)$. Since $f(x_o,\cdot) \in C(Y,Z)$, there exists an open set $U_i \subset Y$ such that $y_o \in U_i$ and $f(x_o,U_i) \subset V_j$. Hence $g(x_o) \in A_{ij}$. Since g is nearly continuous at x_o, the set $g^{-1}(A_{ij}) = \{x \in X : g(x) \in A_{ij}\}$ is dense in some neighborhood W of x_o. This means that $f(x,U_i) \subset V_j$ for all x in some dense subset of W.

This remark and the assumption that $f(\cdot,d) : X \to Z$ is continuous for $d \in D$ (where D is dense in Y) imply that $f(x,d) \in \mathrm{Cl}V_j$ for each fixed $d \in U_i \cap D$ and all $x \in W$.

Now, since Z is regular, we are through by the continuity of $f(x,\cdot)$ and the density of D in U_i. $\square$

5. Characterizations

The first characterization of points of continuity was shown by R. Kershner [Kr]; see also [Gr].

Theorem 5.1 [Ke] Let $S \subset \mathbb{R}^n$. Then S is the set of points of discontinuity of some $f : \mathbb{R}^n \leftrightarrow \mathbb{R}$ if and only if S is an F_σ contained in the product $\prod_{i=1}^{n} A_i$ of sets A_i of first category in R, respectively.

Obviously $C(f) = \mathbb{R}^n \setminus D(f)$.

The following Lemma 5.2 was proved by J. C. Breckenridge and T. Nishiura [BN].

Lemma 5.2 Let A and B be closed, nowhere dense subsets of metric spaces X and Y respectively. If H is a closed subset of $X \times Y$ with $H \subset A \times B$, then there is a function $f: X \times Y \leftrightarrow [0,1]$ such that $f(x,y) = 1$ for $(x,y) \in (A \times Y) \cup (X \times B)$. Furthermore, $D(f) = H = \{(x,y) \in X \times Y : \mathrm{osc}(f(x,y)) = 1\}$.

Sketch of the proof: We first construct a set E contained in $X\times Y$ such that: $E\cap[(A\times Y)\cup(X\times B)] = \emptyset$ and $\mathrm{Cl}\, E\cap[(A\times Y)\cup(X\times B)] = H$. Now we define f by the following formula:

$$f(p) = \begin{cases} 1, & \text{if } p\in(A\times Y)\cup(X\times B) \\ \dfrac{d(p,E)}{d(p,E)+d(p,(A\times Y)\cup(X\times B))}, & \text{otherwise} \end{cases}$$

□

The following Theorem easily follows from [BN].

Theorem 5.3 Let X and Y be compact metric. Further, let M be metric and let $S\subset X\times Y$. Then S is the set of points of discontinuity of some $f: X\times Y \to M$ if and only if S is an F_σ contained in the product $A\times B$ of sets A, B of first category in X and Y respectively.

The following problem of mine has been recorded, around 1978, in the Wrocław New Scottish Book as Problem 944.

Problem 5.4 Let X and Y be compact (Hausdorff) spaces. Characterize C(f) for any $f: X\times Y\to\mathbb{R}$.

6. Namioka and co-Namioka spaces

Let us consider the following general statement which is a special case of our Uniformization Problem formulated in Introduction.

(*) Given any function $f: X\times Y\to Z$, then there is a dense G_δ subset A of X such that $A\times Y\subset C(f)$.

In 1974 I. Namioka showed

Theorem 6.1 Let X be regular, strongly countably complete, Y be locally compact and σ-compact and let Z be pseudo-metric. Then (*) holds.1)

His excellent article [Na] brings many interesting applications (some

1) This result is one of the first results of this type where both X and Y do not have to be metrizable nor satisfy any countability axioms.

Explanation of Diagram 1

Given a function $f\colon X \times Y \to Z$, we shall say that f is *q-weakly separately continuous* (resp. *a.e.-weakly separately continuous*) if:

(1) For each $x \in X$, f_x is continuous

(2) For each $y \in D$, f_y is quasi-continuous (resp., a.e. continuous (in the sense of category)) for some $D \subset \mathrm{Cl}\, D = Y$.

Further, unless some weaker assumptions on a function $f : X \times Y \to Z$ are imposed, recall that

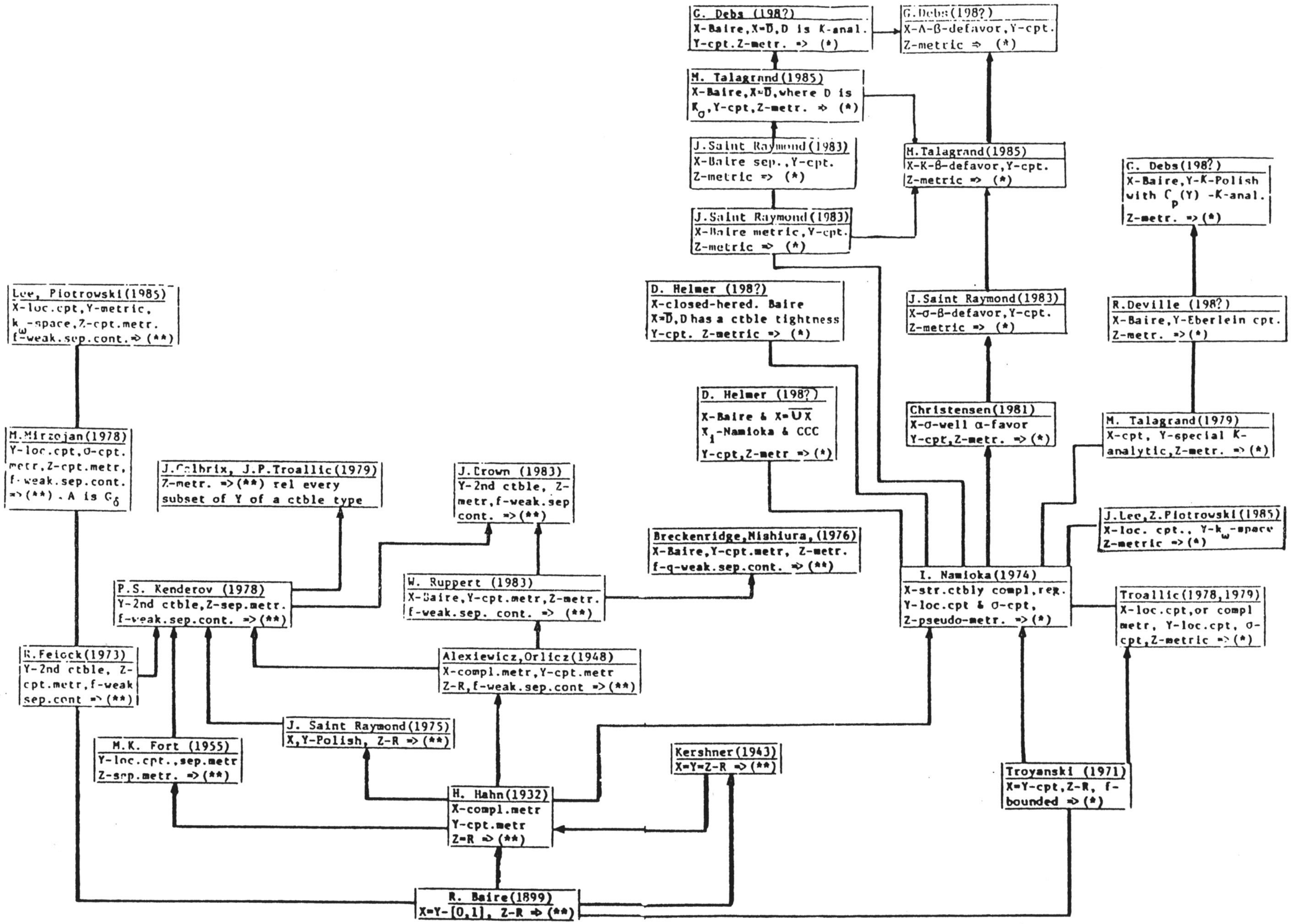
(*) stands for: "For any function $f\colon X \times Y \hookrightarrow Z$ there is a dense G_δ subset A of X such that $A \times Y \subset C(f)$".

(**) stands for: "For any function $f : X \times Y \hookrightarrow Z$ there is a residual set A in X such that $A \times Y \subset C(f)$".

(**) rel B stands for: "For any function $f : X \times Y \hookrightarrow Z$ there is a residual set A in X such that $A \times B \subset C(f)$, where $B \subset Y$".

$\boxed{A} \longrightarrow \boxed{B}$ denotes "B generalizes A"

$\boxed{A} \text{———} \boxed{B}$ denotes "B is related to A"



G. Debs (198?)
X-Baire, X=D̄, D is K-anal.
Y-cpt. Z-metr. => (*)
G. Debs (198?)
X-A-β-defavor, Y-cpt.
Z-metric => (*)
M. Talagrand (1985)
X-Baire, X=D̄, where D is
K_σ, Y-cpt, Z-metr. => (*)
J. Saint Raymond (1983)
X-Baire sep., Y-cpt.
Z-metric => (*)
M. Talagrand (1985)
X-K-β-defavor, Y-cpt.
Z-metric => (*)
G. Debs (198?)
X-Baire, Y-K-Polish
with $C_p(Y)$ -K-anal.
Z-metr. => (*)
J. Saint Raymond (1983)
X-Baire metric, Y-cpt.
Z-metric => (*)
Lee, Piotrowski (1985)
X-loc.cpt, Y-metric,
k_ω-space, Z-cpt.metr.
f-weak.sep.cont. => (**)
D. Helmer (198?)
X-closed-hered. Baire
X=D̄, D has a ctble tightness
Y-cpt. Z-metric => (*)
J. Saint Raymond (1983)
X-σ-β-defavor, Y-cpt.
Z-metric => (*)
R. Deville (198?)
X-Baire, Y-Eberlein cpt.
Z-metr. => (*)
D. Helmer (198?)
X-Baire & $X=\overline{\cup X}$
X_i-Namioka & CCC
Y-cpt, Z-metr => (*)
Christensen (1981)
X-σ-well α-favor
Y-cpt, Z-metr. => (*)
M. Talagrand (1979)
X-cpt, Y-special K-
analytic, Z-metr. => (*)
M. Mirzojan (1978)
Y-loc.cpt, σ-cpt.
metr, Z-cpt.metr,
f-weak.sep.cont.
=> (**). A is G_δ
J. Calbrix, J.P. Troallic (1979)
Z-metr. => (**) rel every
subset of Y of a ctble type
J. Crown (1983)
Y-2nd ctble, Z-
metr, f-weak.sep
cont. => (**)
J. Lee, Z. Piotrowski (1985)
X-loc. cpt., Y-k_ω-space
Z-metric => (*)
Breckenridge, Nishiura, (1976)
X-Baire, Y-cpt.metr, Z-metr.
f-q-weak.sep.cont. => (**)
I. Namioka (1974)
X-str.ctbly compl, reg.
Y-loc.cpt & σ-cpt,
Z-pseudo-metr. => (*)
P.S. Kenderov (1978)
Y-2nd ctble, Z-sep.metr.
f-weak.sep.cont. => (**)
W. Ruppert (1983)
X-Baire, Y-cpt.metr, Z-metr.
f-weak.sep. cont. => (**)
Troallic (1978, 1979)
X-loc.cpt, or compl
metr, Y-loc.cpt, σ-
cpt, Z-metric => (*)
R. Feiock (1973)
Y-2nd ctble, Z-
cpt.metr, f-weak
sep.cont => (**)
Alexiewicz, Orlicz (1948)
X-compl.metr, Y-cpt.metr
Z-R, f-weak.sep.cont => (**)
J. Saint Raymond (1975)
X, Y-Polish, Z-R => (**)
M.K. Fort (1955)
Y-loc.cpt., sep.metr
Z-sep.metr. => (**)
Kershner (1943)
X=Y=Z-R => (**)
Troyanski (1971)
X=Y-cpt, Z-R, f-
bounded => (*)
H. Hahn (1932)
X-compl.metr
Y-cpt.metr
Z=R => (**)
R. Baire (1899)
X=Y-[0,1], Z-R => (**)

Diagram 1

of them will be mentioned in Section 7) and no doubt initiated the renaissance of the topic of separate and joint continuity. Soon after, a group of analysts/topologists "joined the race", including mainly specialists from the famous French School of Mathematics. The question is: "How 'far' can one go in answering (*)?" That is, what kinds of spaces may be assumed as X or Y?

It soon became clear that "the candidates" for X are various topologically complete spaces, while "candidates" for Y are various compact-like spaces.

In fact I. Namioka, anticipating this observation, asked if Theorem 6.1 is true for *any* Baire space X[1)].

J.P.R. Christensen [Cr1] calls a space X *Namioka* if (*) holds for any compact space Y and any metric space Z.[2)]

Quite recently J. Saint Raymond [SR2] showed the following results.

Theorem 6.2 (1) Separable Baire spaces are Namioka.

(2) Tychonoff Namioka spaces are Baire.

(3) In the class of *metric* spaces:

X is Namioka if and only if it is Baire.

In order to proceed further with the presentation of the results, we need the definitions of some spaces in terms of games.

Let X be a space and let α and β be two players with β the first to move. Consider the following games.

(i) Each player chooses a nonempty open set V in X, lying in the opponent's previously chosen open set. α wins if he can choose his V_i sets so that $\bigcap_{i=1}^{\infty} V_i \neq \emptyset$.

1) An answer, due to M. Talagrand [Ta2], came very recently; see Example 6.6.

2) It was shown in [Cr1] that a metric space Z in this definition can be replaced by the unit interval.

(ii) same as (i) except a point is chosen by β in sets chosen by β and open set chosen by α must contain the point. α wins if he can choose his V_i sets so that $\bigcap_{i=1}^{\infty} V_i \neq \emptyset$.

(iii) β starts by choosing an open nonempty subset $U_1 \subset X$. Then α chooses an open subset $V_1 \subset U_1$ and a point $x_1 \in V_1$. β then chooses an open nonempty subset $U_2 \subset V_1$ (he may choose as he wishes but is expected to escape from x_1). Next α chooses an open subset $V_2 \subset U_2$ and a point $x_2 \in V_2$, and so on. α wins if any subsequence $\{x_{n_p}\}$ of the sequence $\{x_n\}$ accumulates to at least one point of the set $\bigcap_{i=1}^{\infty} V_i = \bigcap_{i=1}^{\infty} U_i$.

(iv) same as in (ii), except for the fact that α chooses open subsets $V_i \subset U_i$ and *compact* subsets $K_i \subset V_i$ (rather than points $x_i \in V_i$), where $i = 1,2,\ldots$. α wins if the set $\mathrm{Cl} \bigcap_{i=1}^{\infty} K_i \cap \bigcap_{j=1}^{\infty} V_j \neq \emptyset$.

(v) same as in (iv), except for the sets K_i, chosen by α, are now *K-analytic* (instead of compact).

Now, a space X is called α-favorable (resp.: strongly α-favorable; σ-well α-favorable; K-α-favorable; A-α-favorable) if α has a winning strategy in the game (i) (in the game (ii), (iii) (iv) and (v), respectively).

Further, a space X is called β-defavorable (resp.: σ-β-defavorable; K-β-defavorable; A-β-defavorable), if β does not have any winning strategy in the game (i) (resp. in the game (iii); in the game (iv); in the game (v))[1].

Theorem 6.3 [Cr1] σ-well α-favorable spaces are Namioka.

Two years later, in 1983, J. Saint Raymond improved Theorem 6.3 showing

Theorem 6.4 [Sr2] σ-β-defavorable spaces are Namioka.

---

1) Since there are spaces in which, for example, (i) is not determined, there are β-defavorable spaces which are not α-favorable and so on.

Shortly after M. Talagrand [Ta2] showed that all K-β-defavorable spaces are Namioka. It was shown [Dv] that the class of K-β-defavorable spaces captures important classes of spaces such as Baire metrizable or Baire spaces having dense K_σ subspaces.

Subsequently, G. Debs [Db2] showed that the class of Namioka spaces contains all Baire spaces having dense $\mathcal{K}$-analytic subspaces.

Finally in [Db2] all the mentioned results starting from Theorem 6.3 were taken by

Theorem 6.5 [Db2] A-β-defavorable spaces are Namioka.

And when everything looked like the next class of Namioka spaces are α-favorable ones, M. Talagrand showed

Example 6.6 [Ta2] There exists an α-favorable[1)] space X which is *not* Namioka.

Proof: Let S be an uncountable set and let $2 = \{0,1\}$. Define $X = \{x \in 2^S : |\{s \in S : x(s) = 1\}| \leq \aleph_0\}$. For each $x \in X$ and a countable subset $A \subset S$ define

$$W(x,A) = \{y \in X : \forall s \in A,\ y(s) = x(s)\}.$$

Then

$$\{W(x,A) : x \in X \text{ and } A \text{ is a countable subset of } S\}$$

is a base for a topology on X. It can be shown that X is α-favorable. So, if $Y = \beta S$, then $f : X \times Y \to [0,1]$ given by $f(x,y) = x(y)$ is a function for which the conclusion of (*) does not hold. □

Quite recently R. A. McCoy [Mc] remarked that the space X in Example 6.6 is also pseudo-complete[2)].

---

1) Hence a Baire space.
2) He also showed that X is 0-dimensional Hausdorff (hence Tychonoff); however, it is neither Lindelöf nor satisfies CCC. An interesting question which he raised [Mc] is whether X is normal.

Various complete spaces as Namioka spaces

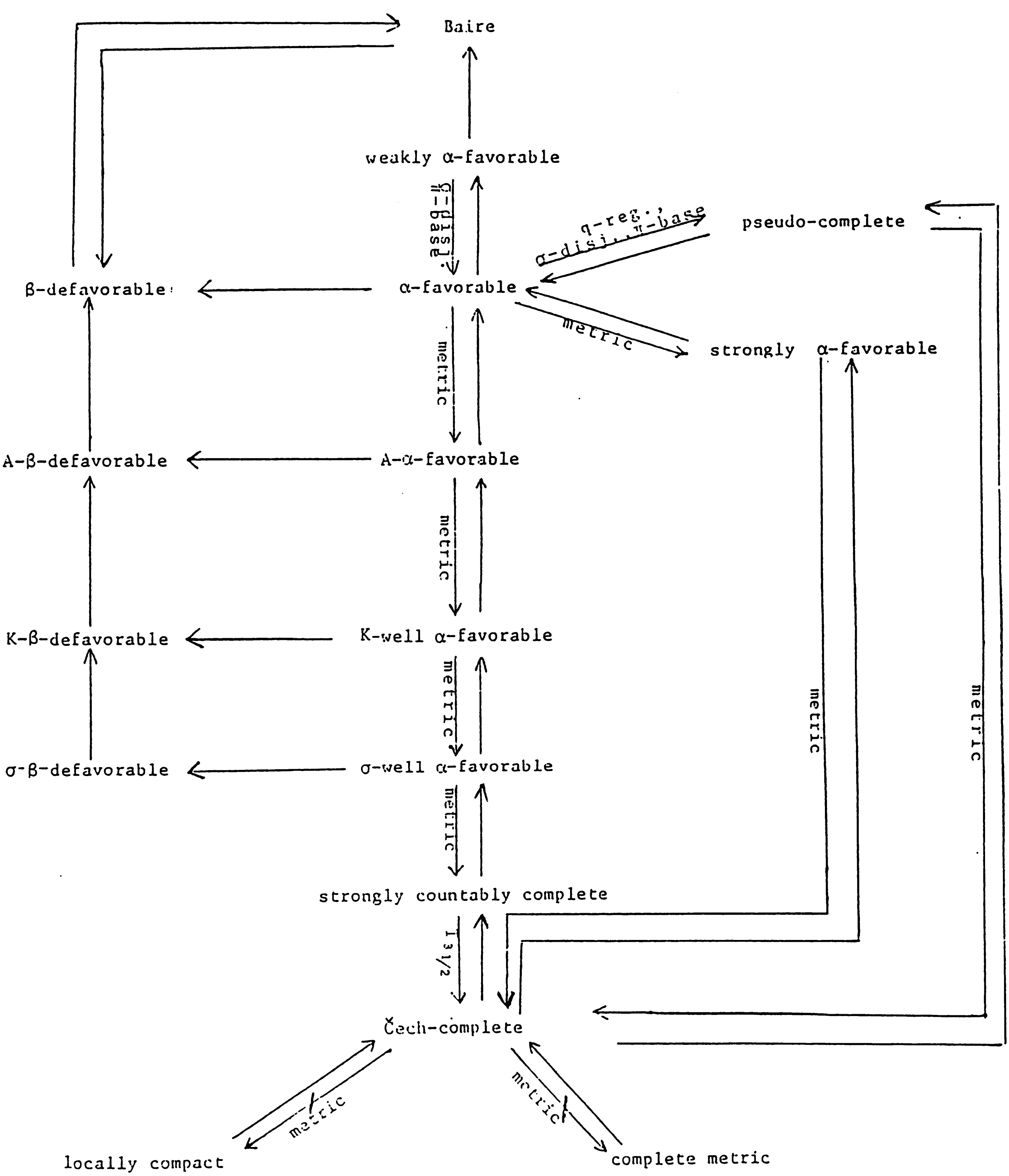


Diagram 2

There is yet another result on Namioka spaces; namely D. Helmer ([Hr2], p. 16) announced that a closed-hereditarily Baire space having a dense subspace of countable tightness is Namioka[1]. Also, another "structural result" of D. Helmer (see also [Hr2], p.16) is of real interest; namely every Baire which contains a sequence of subspaces being Namioka, and satisfying the countable chain condition whose union is dense is also Namioka.

For a mapping approach to Namioka spaces see [HJT].

Let us recall that *Sorgenfrey line* is α-favorable but not σ-well α-favorable [Cr1]. It was stated in [Cr3] that it <u>is</u> Namioka. This fact, however, can also be deduced from the just mentioned Helmer's result ([Hr2], p. 16). In fact, Sorgenfrey line is closed-hereditarily Baire and as hereditarily separable it has countable tightness.

Observe that the function f defined in Example 6.6 still has "many" points of continuity. This fact prompted M. Talagrand to ask the following

<u>Problem 6.7</u> [Ta2] Let X be Baire, Y be compact (Hausdorff) and let $f : X \times Y \to \mathbb{R}$. Is $C(f) \neq \emptyset$?

In an attempt to find a suitable class of spaces Y such that for any Namioka space X and any metric space M the statement (*) is true, the following class of spaces has been defined in [LeP1].

Let S be a "nice" subclass of Namioka spaces (e.g. compact spaces). A space Y is called <u>*co-Namioka*</u>[2] (resp. <u>*co-Namioka rel S*</u>) if for any Namioka

1) Prof. R.W. Hansell has kindly informed me that he has obtained this result independently.

2) Recently G. Debs [Db2] used the term *co-Namioka* spaces for the class of spaces Y, such that (*) holds for any *Baire* space X and any metric space Z.

space X (resp. any space X from S), for any metric space M and any function $f: X \times Y \to M$ there is a dense G_δ set A such that $A \times Y \subset C(f)$.

In the process of showing his main result of [Na], I. Namioka proved

Theorem 6.8 Every locally compact σ-compact space is co-Namioka.

In 1979, M. Talagrand [Tal] showed

Theorem 6.9 Special K-analytic spaces[1)] are co-Namioka rel C, where C stands for the class of compact spaces.

Last year, J. P. Lee and myself [LeP1] have shown

Theorem 6.10 k_ω spaces are co-Namioka rel LC, where LC stands for the class of locally compact spaces.

Further, the following result may be deduced from [CT].

Theorem 6.11 Every second countable space is co-Namioka.

This means, in particular, that if $X = [0,1]$, Y is the set Q of rational numbers and M is metric, then the conclusion of (*) is true!

However, one cannot have the rationals Q as the first factor of the product $X \times Y$. In fact, we have

Example 6.12 [Cr3] Let X = Q, Y = [-1,1] and let $Z = C_p(Q^2, [-1,1])$, the space of continuous functions from Q^2 into [-1,1] with the topology of the pointwise convergence; which is a compact metric space (!). Then there is a function $f: X \times Y \to Z$ for which the conclusion of (*) does not hold.

So, are all Lindelöf spaces co-Namioka?

Example 6.13 [Tal] Let X and Z be the unit interval I and let Y be the space $C_p(I,I)$ of continuous functions from I into I, equipped with the topology of the pointwise convergence. Then $f(x,y) = y(x)$ is a separately continuous

1) See [Tal] for the definition of special K-analytic spaces.

Let X be compact.

Spaces Y for which every separately continuous function f: $X \times Y \to Z$ satisfies (*)

$$\begin{tikzcd}
\text{locally compact} & & \mathcal{K}\text{-analytic} & \\
 & & \text{special } \mathcal{K}\text{-analytic} \arrow[u] & \\
\text{locally compact \& } \sigma\text{-compact} \arrow[uu] \arrow[urr] & & & \text{Polish} \arrow[ul] \\
\text{compact} \arrow[u] & & &
\end{tikzcd}$$

Diagram 3

Let X be Namioka.

Spaces Y for which every separately continuous function f: $X \times Y \to Z$ satisfies (*)

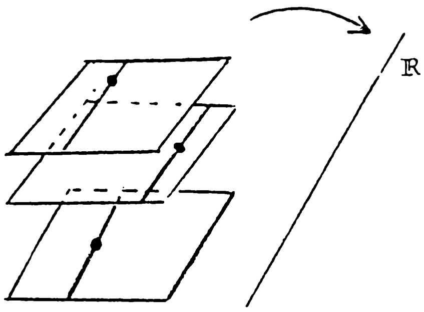
$$\begin{tikzcd}
\check{\text{C}}\text{ech-complete} & & \text{k-space} & \sigma\text{-compact} \\
 & & k_\omega\text{-space (= hemicompact k-space)} \arrow[u] \arrow[ur] & \\
 & \text{locally compact \& } \sigma\text{-compact} \arrow[uul] \arrow[ur] & & \\
 & \text{compact} \arrow[u] & &
\end{tikzcd}$$

Diagram 4

function which does not satisfy the conclusion of (*). So maybe all locally compact spaces that are paracompact, or all k-spaces, are co-Namioka? In particular, does (*) hold for both X and Y being complete metric spaces and Z being metric[1]?

The following example, due to J. B. Brown, answers these questions in the negative.

Example 6.14 ([Bw2]; see also [LePl].)



Let $X = [0,1]$, $Y = \dot{\bigcup}_{\alpha \varepsilon [0,1]} Y_\alpha, Y_\alpha = [0,1]$ ($\dot{\bigcup}$ denotes the free union of Y_α's). Let $f: X \times Y \to \mathbb{R}$ be defined to be separately continuous on every square $X \times Y_\alpha$ and to have a point of discontinuity along the line $x = \alpha$.

Let us close this section with the following two problems.

Problem 6.15 [Ta2] What compact spaces Y are such that for every Baire space X and every $f: X \times Y \to \mathbb{R}$ the conclusion of (*) holds?

Problem 6.16 [LePl] Characterize co-Namioka spaces.

7. Applications.

For the reader's convenience we shall separately list some applications[2] of Namioka-type theorems to topological groups and semigroups and applications to the theory of Banach spaces.

a) Topological groups and semigroups.[3]

---

1) This question has been asked explicitly in [Crl], and implicitly in [AO].
2) Usually these results depend essentially upon particular theorems on separate versus joint continuity; that is, the latter are being applied on a piecemeal basis.
3) Professor N. Brand has informed me that the proofs of a few results regarding topological groups and related to separate and joint continuity are incorrect, namely [Hul], §9, Cor. 3, [Hu2], Chapter II, §17, Cor. 3, p. 38 and [Wu], p. 453; see [Bdl], p. 54 for more information.

— Ellis' theorem on separately continuous actions of locally compact groups on locally compact spaces;see [Na],[Tr2] and [HT].

— compact semitopological semigroups (with identities) acting on compact spaces; see [Lw1], [Lw2], [Hr3] and [HT].

— Ryll-Nardzewski's theorem on minimal ideals of compact semitopological semigroups having dense subgroups of units; see [Tr2],[Rp].

— Corson-Glicksberg theorem on compact subsets of the space of all continuous homomorphisms of a topologically complete group into a topological group; see [Na], [Cr1].

Now, let us list some applications of the results on separate versus joint continuity into Banach spaces.

b) <u>The theory of Banach spaces</u>.

— Troyanski's theorem: weak-compact convex subset of a Banach space is the closed convex hull of its "denting points"; [Na].

— existence of "thick" sets where each continuous convex function from a Banach space is Gateaux differentiable; [St1], [Db1], [LW].

— Johnson's theorem on the norm separability of the range of certain functions;[Cr2].

— first class selectors for weakly upper semi-continuous multivalued maps in Banach spaces; [HJT].

— Radon-Nikodym Property; [ChK], [Ta1].

— compact spaces that are homeomorphic with weakly compact sets in Banach spaces (≡ Eberlein-compacts); [Hr2], [Dv] and [Ta2].

The author would like to thank the referee for his comments which essentially improved the exposition of the results.

I also would like to thank Mrs. K. DeMatteo, the typist, for her patience and understanding.

[AO] Alexiewicz, A., Orlicz, W., Sur la continuité et la classification de Baire des fonctions abstraites, Fund. Math., 35(1948), 105-126. M.R. 10,307

[An] Antosik, P., Study of the continuity of a function of many variables (Russian). Prace Mat. 10(1966) 101-104. M.R. 32 #5809

[Ba] Baire, R., Sur les fonctions des variables réelles, Ann. Mat. Pura Appl. 3(1899), 1-122

[BJM] Berglund, J.F., Junghenn, H.D., Miles, P., Compact right topological semigroups and generalization of almost periodicity. Lecture Notes in Mathematics no. 663, Springer-Verlag, New York 1978 pp. 4-5. M.R. 80 #22003

[Bg1] Bögel, K., Über partiell differenzierbare Funktionen Math. Z., 25(1926). 490-498

[Bg2] __________, Über die Stetigkeit und die Schwankung von Funktionen zweier reeller Veränderlichen, Math. Ann. 81(1920), 64-93.

[Bu] Bourbaki, N., General topology vol. II, Addison-Wesley Publishing Company Reading, Mass. 1966 pp. 255,256 MR 34 #5044b

[Bd1] Brand, N., Über die Stetigkeit der Inversen, Ph.D. Thesis, Münich, 1981 p.54.

[Bd2] __________, Another note on the continuity of the inverse Archiv der Math. 39(1982) 241-245 MR 84 #22001

[BN] Breckenridge, J.C., Nishiura, T., Partial continuity, quasicontinuity and Baire spaces. Bull. Inst. Math. Acad. Sinica 4 (1976), no.2, 191-203 MR 58 #24174

[Bw1] Brown, J. B., Oral communication, Fall 1983.

[Bw2] __________, Oral communication, Spring 1984.

[CT] Calbrix, J., Troallic, J. P., Applications séparément continues C.R. Acad. Sci. Paris 288(1979) 647-648. MR 80 # 54009

[Ca] Carroll, F.W., Separately continuous functions are Baire functions. Amer. Math Monthly 78(1971), 175

[CaK] __________, Koehl, F.S., Separately continuous Banach-valued functions on compact groups. J. London Math. Soc. 4(1971) 100-102: MR 45 #3637.

[Cr1] Christensen, J.P.R., Joint continuity of separately continuous functions. Proc. Amer. Math Soc. 82(1981) 455-461. MR 82 #54012.

[Cr2] ______________________, Theorems of Namioka and R. E. Johnson type for upper semicontinuous and compact-valued set-valued mappings. Proc. Amer. Math. Soc. 86(1982) 649-655. MR 83 #54014.

[Cr3] ______________________, Remarks on Namioka spaces and R.E. Johnson's theorem on the norm separability of the range of certain mappings. Math. Scand. 52(1983) 112-116. MR 85 #46003.

[ChK] ______________________, Kenderov, P.S., Dense strong continuity of mappings and the Radon-Nikodym property. Math. Scand. 54(1984), 70-78.

[Co1] __________________, On sets of maximum indetermination of analytic functions. Math. Z 67(1957), 377-396. MR 20 #2449.

[Co2] __________________, Addendum, Math. Z., 68(1958), 498-499. MR 20 #2450.

[Db1] Debs, G., Gateaux dérivabilité des fonctions convexes continues sur les espaces de Banach non séparables, preprint.

[Db2] ________, Une classe de compacts satisfaisant le théorème de Namioka sur les fonctions séparément continues, preprint.

[Db3] ________, Tactique et stratégie dans le jeu de Choquet, preprint.

[DK] Deely, J.J., Kruse, R.L., Joint continuity of monotonic functions. Amer. Math. Monthly 76(1969) 74-76.

[Dv] Deville, R., Parties faiblement de Baire dans les espaces de Banach, thesis.

[El1] Ellis, R., Locally compact transformation groups. Duke Math. J. 24(1957), 119-125. MR 19,561.

[El2] _________, A note on the continuity of the inverse. Proc. Amer. Math. Soc. 8(1957) 372-373. MR 18, 745.

[Eg] Engelking, R., On Borel sets and B-measurable functions in metric spaces. Annales. Soc. Math. Polon. (Comm. Math.) 10(1967), 145-149. MR35 #66.

[Fk] Feiock, R.E., Cluster sets and joint continuity. J. London Math. Soc. 7(1974), 397-406. MR 50 #5724.

[Ft] Fort Jr., M. K., Category theorems. Fund. Math. 42(1955), 276-288. MR 17, 1115.

[Ge] Genocchi, A., Calcolo differentiale e principii di Calcolo integrale. Torino 1884, pp. 173-174.

[Gl] Glicksberg, I., Weak compactness and separate continuity. Pacific J. Math. 11(1961), 205-214. MR 22 #11275.

[Go] Goffman, C., Coordinate invariance of linear continuity, Arch. Rational. Mech. Anal. 20(1965), 153-162 MR 32,1304.

[Gr] Grande, Z., Une caractérisation des ensembles des points de discontinuité des fonctions linéairement - continues. Proc. Amer. Math. Soc. 52(1975), 257-262. MR 51 #10549.

[Hh1] Hahn, H., Über Funktionen mehrerer Veränderlichen, die nach jeder einzelnen Veränderlichen stetig sind. Math. Z. 4(1919), 306-313.

[Hh2] ________, Reelle Funktionen. Leipzig, 1932 pp. 325-338.

[HJT] Hansell, R.W., Jayne, J.E., Talagrand, M., First class selectors for weakly upper semi-continuous multivalued maps in Banach spaces. J.Reine Angew.Mathematik 361(1985) 201-220.

[HT] Hansel, G., Troallic, J. P., Points de continuité a gauche d'une action de semi-groupe, Semigroup Forum 26(1983) 205-214.

[Hr1] Helmer, D., Joint continuity of affine semigroup actions, Semigroup Forum 21(1980) 153-165. MR 82a #22002.

[Hr2] _________, Criteria for Eberlein compactness in spaces of continuous functions. Manuscripta Math. 35(1981), 27-51 MR 82j #54020.

[Hr3] _________, Continuity of semigroup actions. Semigroup Forum 23(1981), pp. 153-188. MR 83a #22002.

[Hr4] _________, Joint continuity of sequentially continuous maps. Colloq. Math. 46(1982) 167-169. MR 84e #54012.

[Hl] Hill, H.S., Properties of certain aggregate functions. Amer. J. Math. 1927, 419-432.

[Hu1] Husain, T., Semi-topological groups and linear spaces, Math. Annalen 160, (1965), 146-160, MR 32, 4675.

[Hu2] __________, Introduction to topological groups, W.B. Saunders Co., Philadelphia 1966, MR 34, 278.

Jayne, J.E., see Hansell, R.W. [HJT]

[Jh] Johnson, R.E., Separate continuity and measurability. Proc. Amer. Math. Soc. 82(1969) 420-422.

Junghenn, H.D., see Berglund, J.F. [BJM]

[Kp] Kempisty, S., Sur les fonctions quasicontinues. Fund. Math. 19(1932), 184-197.

Kenderov, P. S. see also Christensen, J.P.R. [ChK]

[Ke1] Kenderov,P.S., Dense strong continuity of pointwise continuous mappings. Pacific J. Math. 89(1980), 111-130. MR 82d #46034.

[Ke2] ______________, Multivalued mappings and their properties similar to continuity. Intern. Topol. Conf. 1979, Uspehi Mat. Nauk 35(1980) no. 3(213), 194-196. MR 81i #54013.

[Kr] Kershner, R., The continuity of functions of many variables. Trans. Amer. Math. Soc. 53(1943), 83-100. MR 4, 153.

Koehl, F.S. see Carroll,F. W. [CaK]

[Ko] Kolesnikov, S.V., Characteristics of sets of discontinuities of functions with linearly continuous partial derivatives. Matem. Zametki 25(1979), 40-43. MR 80c #26009.

[KV1] Kolmogorov A., Verčenko, J., On points of discontinuity of functions of two variables (in Russian). Dokl. Akad. Nauk USSR 1(1934), 105-106.

[KV2] ____________________________, A continuation of studies of the points of discontinuity of functions of two variables (in Russian). Dokl. Akad. Nauk USSR 4(1934), 361-362.

Kruse, R.L. see Deely, J.J. [DK]

[Ku1] Kuratowski, K., Sur les fonctions représentables analytiquement et les ensembles de première catégorie. Fund. Math. 5(1923), 75-86.

[Ku2] ______________, Sur la théorie des fonctions dans les espaces métriques. Fund. Math. 17(1931), 275-282.

[Ku3] ______________, Les fonctions semi-continues dans l'espace des ensembles fermés. Fund. Math. 18(1932), 148-159.

[Ku4] ______________, Quelques problèmes concernant les espaces métriques non-séparables. Fund. Math. 25(1935), 534-545.

[LaP] Laczkovich, M. and Petruska, G., Sectionwise properties and measurability of functions of two variables. Acta. Math. Acad. Sci. Hungar 40(1982), 1969-178, MR 84e #26016.

[LW] Lau,K. - S., Weil, C.E., Differentiability via directional derivatives, Proc. Amer. Math. Soc., 70(1978), 11-17. MR 58 #6103

[Lw1] Lawson, J. D., Joint continuity in semitopological semigroups. Illinois J. Math. 18(1974), 275-285, MR 49 #454.

[Lw2] _____________, Additional notes on continuity in semitopological semigroups. Semigroup Forum 12(1976), 265-280, MR 53 # 9175.

[Lb] Lebesgue, H., Sur les fonctious représentable analytiquement. J. Math. Pure Appl. (6) vol.1 (1905) 139-216, cf. 201-202.

[LeP1] Lee, J.P., Piotrowski, Z., A note on spaces related to Namioka spaces. Bull. Austral. Math. Soc. 31 (1985), 285-292.

[LeP2] ________________________, On Kempisty's generalized continuity. Rend. Circ. Mat. Palermo 34(1985).

[Lu] Lusin, N.N., The theory of functions of a real variable (in Russian). Moscow, 1948, pp. 172-180.

[MS] Madison, B.L., Stepp, J.W., Inversion and joint continuity in semigroups on k_ω-spaces. Semigroup Forum 15(1978), 195-198. MR 57 #12759.

[Ml] Mal'seva, N.B., On cluster sets of mappings of topological spaces. Soviet Math. Dokl. 25(1982) 814-817.

[Mr] Marcus, S., Sur les fonctions quasi-continues au sens de S. Kempisty. Colloq. Math. 8(1961), 47-53. MR 23 #3212.

[MR] Marczewski, E., Ryll-Nardzewski, C., Sur la mesurabilité des fonctions de plusieurs variables. Ann. Soc. Polon. Math. 25(1952), 145-154. MR 14, 1070.

[Mt] Martin, N. F. G., Quasi-continuous functions on product spaces. Duke Math. J. 28(1961), 39-44. MR 26 #5419.

[Mc] McCoy, R.A., letter of April 15, 1985

[Mb] Mibu, Y., On quasi-continuous mappings defined on product spaces. Proc. Japan Acad. 192(1958), 189-192. MR 20 # 4613.

Milnes, P., see Berglund, J. F. [BJM]

[Mz] Mirzojan, M.M., On the cluster sets of mappings of topological spaces. Soviet Math. Dokl. vol. 19, no.6(1978) pp. 1326-1329. MR 80f #54005.

[Mg] Montgomery, D., Non-separable metric spaces. Fund. Math. 25(1935), 527-533.

[Mo] Moran, W., Separate continuity and supports of measures. J. London Math. Soc. 44(1969), pp.320-324. MR 38 #4642.

[Mk] Munkers, J. R., Topology, A first course. Prentice Hall Inc. Englewood Cliffs, N.J. (1975) p. 110.

[Na] Namioka, I., Separate continuity and joint continuity. Pacific J. Math. 51(1974), 515-531, MR 51 #6693.

[Nb1] Neubrunn, T., A generalized continuity and product spaces. Math. Slov. 26(1976), 97-99. MR 55 #9015.

[Nb2] ____________, Generalized continuity and separate continuity Math. Slovaca 27(1977), 307-314. MR 80k #54017.

Nishiura, T. see Breckenridge [BN]

Orlicz, B., see Alexiewicz, A. [AO]

Petruska, G., see Laczkovich, M. [LaP]

Piotrowski, Z., see also Lee, J.P. [LeP1] and [LeP2]

[Pt1] Piotrowski, Z., Separate almost continuity and joint continuity. Collog. Math. Soc. J. Bolya Budapest (Hungary) 1978, 957-962. North-Holland, Amsterdam, 1980. MR 82a #54020.

[Pt2] ______________, Continuity points in $\{x\} \times Y$. Bull. Soc. Math. France 108(1980), 113-115. MR 82m #54007.

[Pt3] ______________, Quasi-continuity and product spaces, Proc. Intern. Conf. Geom. Top. (Warsaw 1978), 349-352. PWN, Warsaw, 1980. MR 84d #54023.

[Pt4] ____________, Blumberg property versus almost continuity, preprint

[Pr] Pringsheim, A. Encykl. der Math. Wiss. mit Einschluss ihrer Anwendungen, Bd. II A.1. Leipzig, 1899-1916, footnote 254.

[Rg] Rogers, C.A. et altera, Analytic Sets, Academic Press.London 1980. Part 3, §2, Automatic continuity. MR 82m #03063.

[Rs] Rosenthal, A., On the continuity of functions of several variables. Math. Z. 63(1955), 31-38. MR 17,245.

[Ru] Rudin, W., Principles of Mathematical Analysis, McGraw-Hill Book Co., 1976 p.99.

[Rp] Ruppert, W., On structural methods and results in the theory of compact semitopological semigroups pp. 215-238, from: Lecture Notes in Mathematics no.998. Recent developments in the algebraic, analytical and topological theory of semigroups. Edited by, K.H. Hoffman, H. Jürgensen and H.J. Weinert, Springer-Verlag, 1983. MR 84m #22004

Ryll-Nardzewski, C. see Marczewski, E. [MR]

[SR1] Saint Raymond, J., Fonctions séparément continues sur le produite de deux espaces polonais. Seminaire Choquet (Initiation à l'analyse) 1975/76,communication no.2, 3 p.

[SR2] _______________, Jeux topologiques et espaces de Namioka. Proc. Amer. Math. Soc. 87(1983), 499-504. MR 83m #54060.

[Sl] Slobodnik, S.G., On an expanding system of linearly closed sets. Matem. Zametki 19(1976), 67-84. MR 53 #13494.

[St1] Stegall, C., Gateaux differentiation of functions on a certain class of Banach spaces, Functional Analysis: Surveys and Recent Results III, K. D. Bierstedt, B. Fuchssteiner, Elsevier Science Publishers, B.V., 1984, 35-46.

[St2] __________, A result concerning the weak topology generated by the extreme linear functionals; to appear in Funct. Analysis Seminar, Univ. Paris VII. 1983-1984

Stepp, J. W. see Madison, B.L., [MS]

Talagrand, M., see also Hansell, R. W. [HJT]

[Ta1] Talagrand, M., Deux generalisations d'un théorème de I. Namioka. Pacific J. Math. 81(1979), 239-251. MR 80k #54018

[Ta2] ____________, Espaces de Baire et espaces de Namioka Math. Ann. 270 (1985), 159-164.

[Ta3] ____________, Espaces K-analytiques et espaces de Baire de fonctions continues, preprint.

[Ta4] ____________, Jeux topologiques et espaces de Banach, preprint.

[Th] Thomae, J., Abriss einer Theorie der complexen Funktionen, 2nd ed. Halle 1873, p.15.

[To] Tolstoff, T., On partial derivatives. Izv. Akad. Nauk SSSR, Ser. Mat. 13(1949), 425-446. Engl. transl. Amer. Math. Soc. Transl. 1(1952), 55-83. MR 11,167.

Troallic, J.P. see also Calbrix, J. [CT] and Hansel, G. [HT].

[Tr1] Troallic, J.P., Fonctions à valeurs dans des espaces fonctionnels généraux: Theorèmes de R. Ellis et de I. Namioka, C.R. Acad. Sci. Paris 287(1978), 63-66. MR 80d #54012.

[Tr2] ______________, Espaces fonctionnels et théorèmes de I. Namioka. Bull. Soc. Math. France 107(1979) 127-137. MR 81d #54035.

[Tr3] ______________, Semigroupes semitopologiques et presque-periodicite, pp. 239-251, from: Lecture notes in Mathematics no. 998, see [Rp].

[Ty] Troyanski, S.L., On locally uniformly convex and differentiable norms in certain nonseparable Banach spaces, Studia Math. 37(1971), 173-180. MR 46 #5995.

[VV] Van Vleck, E. B., A proof of some theorems on pointwise discontinuous functions. Trans. Amer. Math. Soc. 8(1907). 189-204.

Verčenko, J., see Kolmogorov, A., [KV1] and [KV2].

[Vo] Volterra, G., see R. Baire [Ba] p. 95.

Weil, C.E., see Lau, K.-S., [LW].

[We] Weston, J.D., Some theorems on cluster sets J. London Math.Soc. 33, (1958), 435-441. MR 20 #7109.

[Wi] Wilhelm, M., Almost lower semicontinuous multifunctions and the Souslin-graph theorem Comm. Math. Univ. Carolinae 23(1982), 147-158. MR 83f #54016.

[Wu] Wu, T.-S., Continuity in topological groups, Proc. Amer. Math. Soc. 13(1962), 452-453. MR 25, 1234.

[Yo] Young, W.H., A note on monotone functions. Quart. J. Math. Oxford Ser. 41(1910), 79-87.

[YY] ______________, Young, G.C., Discontinuous functions continuous with respect to every straight line. Quart. J. Math. Oxford Ser. 41(1910), 87-93.

[Że] Żelazko, W., Metric generalisations of Banach algebras, Dissertationes Math. 47(1965), Warsaw, MR 33 #1752.