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THE METHOD OF FRACTIONAL OPERATORS APPLIED TO SUMMATION

The purpose of this paper is to show how some interesting results concerning series summation and the psi function are established by means of fractional operators. Our main interest here is the method used to obtain the formula

$$(1)\qquad \psi(\lambda) - \psi(\lambda - \nu) = \frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu + n)}{n\Gamma(\lambda + n)}, \quad \mathrm{Re}(\lambda) > \mathrm{Re}(\nu) \geqq 0.$$

The technique used here to construct relation (1) has been called by the misnomer <u>fractional calculus</u> [1], [2], [3], but we shall refer to it as the method of <u>fractional operators</u>. The great elegance that can be achieved by the proper use of fractional operators should more than justify a more general recognition and use. These operators have the power to simplify the solutions of a complicated functional equations. This paper augments an idea initiated by Ross [4]. We will show that Ross's result is a special case (1) if the parameters ν and λ are appropriately specified.

The integral

$$(2) \qquad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)\,dt\ , \qquad\qquad \mathrm{Re}(\nu) \geqq 0,$$

is called the Riemann-Liouville integral and is of fundamental importance in the fractional calculus. This integral defines differentiation and integration to an arbitrary order. The operator notation which best describes this integral, invented by Harold T. Davis [1], is

$$(3) \qquad {}_0D_x^{-\nu} f(x)\ ,$$

where the subscripts on D are the terminals of integration and ν is arbitrary. For a wide class of functions the integral (2) is a beta integral and is readily evaluated.

We start with the beta integral

$$(4) \qquad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} t^{\lambda-1} dt = \frac{\Gamma(\lambda)}{\Gamma(\lambda+\nu)}\, x^{\lambda+\nu-1}\ ,$$

$\mathrm{Re}(\lambda) > 0$, $\mathrm{Re}(\nu) \geqq 0$.

The integral on the left above is differentiated with respect to the parameter λ according to Leibniz's rule. The right side above is also differentiated with respect to λ getting

$$\frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}(\ln\, t)\, t^{\lambda-1} dt = \frac{d}{d\lambda}\left(\frac{\Gamma(\lambda)\, x^{\lambda+\nu-1}}{\Gamma(\lambda+\nu)}\right)$$

The differentiation of the quotient on the right side above is tedious but straightforward. If we recall the psi function relations [5], [6], $\Gamma'(\lambda)/\Gamma(\lambda) = \psi(\lambda)$ and $\Gamma'(\lambda + \nu)/\Gamma(\lambda + \nu) = \psi(\lambda + \nu)$, we will have

$$(5) \qquad \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} (\ln t) t^{\lambda-1} dt = [\ln x + \psi(\lambda) - \psi(\lambda+\nu)] \frac{\Gamma(\lambda) x^{\lambda+\nu-1}}{\Gamma(\lambda + \nu)}$$

For convenience, designate the right side of (5) by $F(x,\lambda,\nu)$. The left side of (5), by the definition (3), can be written as ${}_0D_x^{-\nu} (x^{\lambda-1} \ln x)$. Then, for integration of $x^{\lambda-1} \ln x$ to an arbitrary order we have

$$(6) \qquad {}_0D_x^{-\nu}(x^{\lambda-1} \ln x) = F(x,\lambda,\nu).$$

Because of the property of analyticity and continuity at $\nu = 0$, we can interchange the roles of $-\nu$ and ν. So, for differentiation of $x^{\nu-1} \ln x$ to an arbitrary order we will have

$$(7) \qquad {}_0D_x^{\nu} (x^{\lambda-1} \ln x) = \left(\ln x + \psi(\lambda) - \psi(\lambda - \nu)\right) \frac{\Gamma(\lambda) x^{\lambda-\nu-1}}{\Gamma(\lambda - \nu)}$$

$$= F(x,\lambda, -\nu) \quad .$$

For a consistency check in the above, let $\nu = 1$ and $\lambda = 2$. The ordinary derivative of the left side

$$(8) \qquad D\ (x \ln x) = 1 + \ln x.$$

For these same values $\nu = 1$ and $\lambda = 2$ in the right side of (7), we get

$$(9) \qquad F(x,\lambda, -\nu) = \left(\ln x + \psi(2) - \psi(1)\right) \frac{\Gamma(2)\ x^0}{\Gamma(1)} \quad .$$

But $\psi(2) = 1 + \psi(1)$ getting $1 + \ln x$ in agreement with (8).

A more interesting consistency check with ν arbitrary is given later.

Now we proceed to solve an integral equation of the Volterra type

$$(10) \qquad \frac{1}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1} f(t)\, dt = x^{\lambda-1} \ln x,$$

$\mathrm{Re}\,(\nu) \geqq 0$, $\mathrm{Re}(\lambda) > 0$.

The above integral is a Riemann-Liouville integral and because it is of the convolution type the equation can also be solved by Laplace transforms. However, solving it by the use of fractional operators will exemplify the power, elegance and simplicity of the notation of the fractional calculus. By the definition (3), Eq. (10) is written as

$${}_0D_x^{-\nu} f(x) = x^{\lambda-1} \ln x.$$

Operating on both sides with ${}_0D_x^{\nu}$ gives

$$f(x) = {}_0D_x^{\nu}\,(x^{\lambda-1} \ln x) \; .$$

The application of the result (7) gives us at once the solution to (10):

$$(11) \qquad f(x) = \Big(\ln x + \psi(\lambda) - \psi(\lambda - \nu)\Big)\, \frac{\Gamma(\lambda)}{\Gamma(\lambda-\nu)}\, x^{\lambda-\nu-1} \; .$$

Following Ross [4], we verify this result by substituting (11) into (10) in terms of the argument t. We write a series expansion for ln t as follows:

$$t = x + t - x = x(1 + \tfrac{t-x}{x}) ,$$

where x and t are real and $x > 0$. Then

$$\ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right) .$$

When $\left|\frac{t-x}{x}\right| < 1$, we can expand $\ln\left(1 + \frac{t-x}{x}\right)$ into a Taylor's series expansion. Thus,

$$\text{(12)} \qquad \ln t = \ln x - \sum_{n=1}^{\infty} \frac{(x-t)^n}{nx^n} ,$$

where the interval of convergence is $0 < t \leq 2x$.

When (12) and (11) are substituted into (10) we will have

$$\text{(13)} \qquad \frac{\Gamma(\lambda)}{\Gamma(\nu)\,\Gamma(\lambda-\nu)} \left[\Big(\ln x + \psi(\lambda) - \psi(\lambda - \nu)\Big) \int_0^x (x-t)^{\nu-1} t^{\lambda-\nu-1} dt \right.$$

$$\left. - \int_0^x (x-t)^{\nu-1} t^{\lambda-\nu-1} \sum_{n=1}^{\infty} \frac{(x-t)^n}{nx^n} dt \right] = x^{\lambda-1} \ln x .$$

The above integrals are beta integrals of the form of (4)

$$\int_0^x (x-t)^d t^a dt = \frac{\Gamma(d+1)\,\Gamma(a+1)}{\Gamma(d + a + 2)} x^{d + a + 1} , \qquad \mathrm{Re}(d) > -1 ,$$

$\mathrm{Re}(a) > -1$.

The first integral in (13) has the value

$$x^{\lambda-1} \ln x + \Big(\psi(\lambda) - \psi(\lambda - \nu)\Big) x^{\lambda-1} .$$

The second integral in (13)

$$-\frac{\Gamma(\lambda)}{\Gamma(\nu)\Gamma(\lambda-\nu)}\left[\int_0^x \frac{(x-t)^{\nu}t^{\lambda-\nu-1}}{x}dt + \int_0^x \frac{(x-t)^{\nu+1}t^{\lambda-\nu-1}}{2x^2}dt + \cdots\right.$$

has the value

$$-\frac{\Gamma(\lambda)}{\Gamma(\nu)}\left[\frac{\Gamma(\nu+1)}{\Gamma(\lambda+2)} + \frac{\Gamma(\nu+1)}{\Gamma(\lambda+2)} + \cdots\right] x^{\lambda-1} .$$

Then for (13) we get

$$x^{\lambda-1}\ln x + \Big(\psi(\lambda) - \psi(\lambda-\nu)\Big)x^{\lambda-1}$$

$$\left. - \frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\right] x^{\lambda-1} = x^{\lambda-1}\ln x .$$

After simplification, we obtain the result

$$\text{(14)}\qquad \psi(\lambda) - \psi(\lambda-\nu) = \frac{\Gamma(\lambda)}{\Gamma(\nu)}\sum_{n=1}^{\infty}\frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}, \ \operatorname{Re}(\lambda) > \operatorname{Re}(\nu) \geqq 0$$

$$\text{(15)}\qquad = \frac{\nu}{\lambda} + \frac{1}{2}\frac{\nu(\nu+1)}{\lambda(\lambda+1)} + \frac{1}{3}\frac{\nu(\nu+1)(\nu+2)}{\lambda(\lambda+1)(\lambda+2)} + \cdots .$$

As was stated earlier in (6), the fractional operator (3) has the property of analyticity and continuity at $\nu = 0$. Thus, in the above, we can replace ν with $-\nu$. Then, after multiplying both sides of the above by -1, we obtain the result

$$(16) \quad \psi(\lambda + \nu) - \psi(\lambda) = \frac{\nu}{\lambda} - \frac{1\nu(\nu - 1)}{2\lambda(\lambda + 1)} + \frac{1}{3}\,\frac{\nu(\nu - 1)(\nu - 2)}{\lambda(\lambda + 1)(\lambda + 2)} - \cdots .$$

The relation (16) was obtained by N.E. Nörlund (1925) [8] by the use of the calculus of finite differences. Further, it is not unreasonable to suppose that (15) can be obtained through the use of hypergeometric identities and the appropriate choice of the parameters. At the risk of being guilty of repetition, we restate that our purpose is not the result obtained but, is instead, to expose the power of the proper use of fractional operators.

Formula (14) leads to some interesting results. If we specialize the parameters letting $\lambda = 1$ and $\nu = 1/2$, we will get

$$\psi(1) - \psi(1/2) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n\Gamma(1 + n)} .$$

Using well-known properties of the gamma function and the psi function, the above becomes

$$\ln 4 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n2^n n!} . \quad [4]$$

If we let $\lambda = 1$ and $\nu = 1/3$, then (14) gives

$$\psi(1) - \psi(1/3) = \frac{1}{\Gamma(1/3)} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{3} + n)}{n\Gamma(1 + n)}$$

$$= \frac{1}{3} + \frac{1\cdot 4}{2\cdot 2!\,3^2} + \frac{1\cdot 4\cdot 7}{3\cdot 3!\,3^3} + \cdots .$$

Noting that $\psi(1) - \psi(2/3) = -\frac{\sqrt{3}\,\pi}{6} + \frac{3}{2} \ln 3$, [6], we get the result

$$(17) \qquad -\frac{\sqrt{3}\pi}{6} + \frac{3}{2} \ln 3 = \sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{n3^n n!} .$$

Many series can be summed in this manner. If we let $\lambda - \nu = 1$ in the generalization (14), we will get a well-known formula for the psi function. We will have for (14)

$$\psi(1 + \nu) - \psi(1) = \frac{\Gamma(1 + \nu)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu + n)}{n\Gamma(\nu + n + 1)},$$

which, after simplification, yields the classical result

$$(18) \qquad \psi(1 + \nu) + \gamma = \sum_{n=1}^{\infty} \frac{\nu}{n(\nu + n)} ,$$

where γ is Euler's constant.

The relation (18) which we obtained by specializing parameters in the generalization (14) is obtained in various texts by methods which are conceptually different than given here, for example, [7].

We step back to Eq. (7) to exemplify an elegant and useful feature of the fractional calculus. A simple consistency check was given in (8) and (9) where ν was an integer. Let us consider a consistency check of (7) with $\nu = 1/2$ and $\lambda = 2$. Eq. (7) then yields the derivative of the product $x \ln x$ to the order 1/2:

$$(19) \qquad {}_0D_x^{\frac{1}{2}} (x \ln x) = \left(\ln x + \psi(2) - \psi(\tfrac{3}{2})\right) \frac{2}{\sqrt{\pi}} x^{1/2} .$$

The recurrence formula for the psi function is

$$\psi(x + 1) = \psi(x) + \frac{1}{x} .$$

Thus

$$\psi(2) = \psi(1) + 1.$$

Some known values of the psi function are

$$\psi(1) = -\gamma, \text{ where } \gamma \text{ is Euler's constant}$$

(20) $$\psi(\tfrac{1}{2}) = - \gamma - \ln 4.$$

(21) $$\psi(\tfrac{3}{2}) = - \gamma - \ln 4 + 2.$$

$$\psi(2) = - \gamma + 1.$$

When the above values are substituted into (19), we get the result

(22) $${}_0D_x^{\frac{1}{2}} (x\ln x) = \frac{2}{\sqrt{\pi}}(\ln x + \ln 4 - 1)x^{\frac{1}{2}} .$$

Now we will apply Leibniz's rule for the derivative of a product to an arbitrary order to the left side of (19) to see if we get the same result as in (22). Leibniz's rule is

$${}_0D_x^{\nu} f(x)g(x) = \sum_{k=0}^{\infty} \binom{\nu}{k} {}_0D_x^{(k)} f(x) \quad {}_0D_x^{(\nu-k)} g(x), \qquad [10]$$

where the generalized binomial coefficient

$$\binom{\nu}{k} = \frac{\Gamma(\nu + 1)}{k!\Gamma(\nu - k + 1)}$$

and where $D^{(k)}f(x)$ is ordinary differentiation and $D^{(\nu-k)}g(x)$ is a differintegration to an arbitrary order.

For $\nu = 1/2$, we have

$$(23)\qquad {}_0D_x^{\frac{1}{2}}\,(x\ln x) = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}\, {}_0D_x^{(k)}x \;\; D^{(\frac{1}{2}-k)}\ln x.$$

We note that the series terminates with k = 2. The generalized binomial coefficients are

$$\binom{\frac{1}{2}}{0} = 1 \quad\text{and}\quad \binom{\frac{1}{2}}{1} = \frac{1}{2}\,.$$

Eq. (23) is then

$$(24)\qquad {}_0D_x^{\frac{1}{2}}\,(x\ln x) = (1)xD^{\frac{1}{2}}\ln x + \frac{1}{2}(1)D^{-\frac{1}{2}}\ln x.$$

From [4] we observe that

$$(25)\qquad \begin{cases} {}_0D_x^{\frac{1}{2}}\ln x = \dfrac{x^{-\frac{1}{2}}}{\sqrt{\pi}}\Big(\ln x - \gamma - \psi(\tfrac{1}{2})\Big) \\ \text{and} \\ {}_0D_x^{-\frac{1}{2}}\ln x = \dfrac{2x^{\frac{1}{2}}}{\sqrt{\pi}}\Big(\ln x - \gamma - \psi(\tfrac{3}{2})\Big). \end{cases}$$

Replacing the values of $\psi(\frac{1}{2})$ and $\psi(\frac{3}{2})$ in the above with (20) and (21) and then putting (25) into (24), we will have

$$\begin{aligned} {}_0D_x^{\frac{1}{2}}\,(x\ln x) &= \frac{x^{\frac{1}{2}}}{\sqrt{\pi}}(\ln x + \ln 4) + \frac{x^{\frac{1}{2}}}{\sqrt{\pi}}\Big(\ln x + \ln 4 - 2\Big) \\ &= \frac{2}{\sqrt{\pi}}\,(\ln x + \ln 4 - 1)x^{\frac{1}{2}} \end{aligned}$$

which is in agreement with (22).

It might be of interest to show that the series (17) can be summed by other means. This method is outlined below and the reader can compare it with the method used to obtain (17). We observe that the binomial expansion for $|x| < 1$ of

$$(1-x)^{-1/3} = 1 + \sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n n!} x^n .$$

Divide both sides of the above by x and integrate from 0 to 1 getting

$$\int_0^1 \frac{(1-x)^{-1/3} - 1}{x}\, dx = \sum_{n=1}^{\infty} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{n3^n n!} .$$

So, if we can evaluate the formidable integral above, we can determine the sum of the series. Because this type of integral is rarely seen in applications, it may be worthwhile to sketch the steps. The integral is rationalized by letting $u = (1-x)^{-1/3}$. We have, after simplifications and partial fraction decomposition,

$$3\int_1^{\infty} \frac{1}{u(u^2+u+1)}\, du = 3\int_1^{\infty} \left[\frac{1}{u} - \frac{1}{2}\cdot\frac{2u+1+1}{u^2+u+1}\right] du$$

$$= 3\int_1^{\infty} \left[\frac{1}{2} - \frac{1}{2}\cdot\frac{2u+1}{u^2+u+1} - \frac{1}{2}\cdot\frac{1}{u^2+u+\frac{1}{4}+\frac{3}{4}}\right] du$$

$$= 3\left[\ln\frac{u}{(u^2+u+1)^{\frac{1}{2}}} - \frac{i}{2\sqrt{3}} \arctan\frac{u+1/2}{3/2}\right]_1^{\infty}$$

$$= \frac{3}{2}\ln 3 - \frac{\sqrt{3}\,\pi}{6} ,$$

which is in agreement with (17).

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