Real Analysis Exchange Vol. 11 (1985-86)

Richard G. Gibson, Department of Mathematical Sciences, Columbus College, Columbus, Georgia, 31993; and Fred Roush, Department of Mathematics, Alabama State University, Montgomery, Alabama, 36101.

THE UNIFORM LIMIT OF CONNECTIVITY FUNCTIONS

A. Lindenbaum [4] showed that any real-valued function defined on an interval is the pointwise limit of a sequence of Darboux functions. Phillips [5] obtained the same results with a sequence of functions whose graph is connected and Kellum [3] showed that the same is true for almost continuous functions. In the paper by Kellum an example was given which showed that the uniform limit of a sequence of almost continuous functions $R \to R$ need not be a Darboux function where R is the set of real numbers. Thus the uniform limit of a sequence of connectivity functions need not be a Darboux function. However it must belong to the class of functions characterized by Bruckner, et. al., [2].

Now a natural question arises: "Does there exist a Darboux function which is not the uniform limit of a sequence of connectivity functions?" In this paper we construct a Darboux function $f: I \to I$ which is not the uniform limit of a sequence of connectivity functions where $I = [0,1]$. We also prove the following propositions.

<u>Proposition A</u>. Let X be a metric space. Then the uniform limit f of a sequence $f_m: X \to R$ of peripherally continuous functions is peripherally continuous.

The following propositions follow as corollaries from proposition A.

<u>Proposition B</u>. Let $\{f_m\}$ be a sequence of real-valued functions defined on an interval. If each f_m is a Baire class 1 Darboux function and f_m converges to f uniformly, then f is a Baire class 1 Darboux function, [1].

<u>Proposition C</u>. Let $\{f_m\}$ be a sequence of functions such that each $f_m: I^n \rightarrow I$ where $n \geq 2$. If each f_m is a connectivity function and f_m converges to f uniformly, then f is a connectivity function.

Let X and Y be topological spaces and let $f: X \rightarrow Y$ be a function. Then f is said to be an almost continuous function provided that if U is an open subset of $X \times Y$ containing the graph of f, then U contains the graph of a continuous function with the same domain. f is said to be a connectivity function provided that if C is a connected subset of X, then the graph of f restricted to C is a connected subset of $X \times Y$. f is said to be a Darboux function if $f(C)$ is connected in Y whenever C is connected in X. f is said to be peripherally continuous provided that for any $x \in X$ and any open sets U and V containing x and $f(x)$, respectively, there exists an open set W such that $x \in W \subset U$ and $f(bd(W)) \subset V$ where $bd(W)$ is the boundary of W.

For real-valued functions defined on an interval, almost continuous functions are connectivity functions, connectivity functions are Darboux functions, and functions with connected graphs are connectivity functions. A real-valued function f defined on an interval is peripherally continuous if and only if for each x, there exist sequences $x_m \uparrow x$ and $y_m \downarrow x$ such that $f(x) = \lim f(x_m) = \lim f(y_m)$, [6]. Also a Darboux function has this property, and hence it is a peripherally continuous function, [1], [6].

Example. Let $C \subset I$ be the standard Cantor set. Let U_1 be the set of open intervals removed at odd steps and let U_2 be the set of open intervals removed at even steps. In an interval J of U_1 define f to be any Darboux function with graph dense in $J \times [0, .7]$. In an interval J of U_2 define f to be any Darboux function with graph dense in $J \times [.3, 1]$. On the endpoints of intervals in U_1 let the value of f be 0.2 and on the endpoints of intervals in U_2 let the value of f be 0.8. Elsewhere let f have a value of 0. Thus f is a Darboux function. Since f can be arbitrary on the points of C which are not endpoints of the open intervals removed, there are 2^c of these Darboux functions.

Let K be the closed set defined as follows.

$K = (C \times [.25, .75]) \cup (\{J \times \{.75\} : J \in U_1\}) \cup (\{J \times \{.25\} : J \in U_2\})$

Then K separates the graph of f as well as the graph of any other function within a vertical distance 0.05 of f. Thus f is not the uniform limit of a sequence of connectivity functions $I \to I$.

Proof of proposition A. Choose any $x \in X$ and let $\varepsilon > 0$. Let U be an open set containing x and V be the open interval of radius $\frac{1}{2}\varepsilon$ about $f(x)$. Let N be such that if $m \geq N$, then $|f_m(t) - f(t)| < \frac{1}{2}\varepsilon$ for any $t \in X$. In particular $f_m(x) \in V$ for any $m \geq N$. Let $k \geq N$. Since f_k is peripherally continuous and $f_k(x) \in V$, there exists an open set W such that $x \in W \subset U$ and $f_k(bd(W) \subset V$. So $|f_k(z) - f(x)| < \frac{1}{2}\varepsilon$ for all $z \in bd(W)$.

Choose any $y \in f(bd(W))$. Then there exists $z \in bd(W)$ such that $f(z) = y$. Now

$$\begin{aligned} |y - f(x)| &= |f(z) - f(x)| \\ &= |f(z) - f_k(z) + f_k(z) - f(x)| \\ &\leq |f(z) - f_k(z)| + |f_k(z) - f(x)| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon \\ &= \varepsilon. \end{aligned}$$

Thus $y \in (f(x) - \varepsilon, f(x) + \varepsilon)$, and hence $f(bd(W))$ is a subset of $(f(x) - \varepsilon, f(x) + \varepsilon)$.

Proof of proposition B. Since the uniform limit of Baire class 1 functions is a Baire class 1 function, the result follows from proposition A.

Proof of proposition C. Since on n-cells, $n \geq 2$, there is no distinction among connectivity functions and peripherally continuous functions, the results follows from proposition A.

The function $f: I \to I$ defined by $f(x) = 0$, if x is rational; and $f(x) = 1$, if x is irrational, is a peripherally continuous function which is not the uniform limit of a sequence of Darboux functions. Thus to complete the set of relations among almost continuous, connectivity, Darboux, and peripherally continuous functions and their uniform limits leads to the following question.

Question. Does there exist a connectivity function $f:I \to I$ that is not the uniform limit of a sequence of almost continuous functions $f_m:I \to I$?

Referee's Remark. The reader should also see the paper, "On jumping functions by connected sets", Czech. Math. J. 22(1972), 435-448, by A. M. Bruckner and J. Ceder, where all possible inclusion (or non-inclusion) relationships between the function classes C (continuous), K (connected), U (weakly connected), and D (Darboux), and their uniform closures, $\bar{C}$, $\bar{K}$, $\bar{U}$, and $\bar{D}$, respectively are worked out except for the questions of whether $D \subset \bar{K}$ (settled in this paper), $U \subset \bar{K}$, and $\bar{K} \subset U$.

References

1. A. M. Bruckner, Differentiation of Real Functions, Lecture Notes in Mathematics 659, Springer-Verlag, 1978.

2. A. M. Bruckner, J. G. Ceder, and Max Weiss, "Uniform limits of Darboux functions," Colloquium Math. 15(1966), 65-77.

3. K. R. Kellum, "Sums and limits of almost continuous functions," Colloquium Math. 31(1974), 125-128.

4. A. Lindenbaum, "Sur quelques proprietis des fonctions de variable reele," Ann. Soc. Polon. Math. 6(1927), 129.

5. D. Phillips, "Real functions having graphs connected and dense in the plane," Fund. Math. 75(1972), 47-49.

6. J. Young, "A theorem in the theory of functions of a real variable," Rend. Circ. Math. Palermo 24(1907), 187-192.

Received March 12, 1985