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A MOMENT INEQUALITY

1. Introduction.

In his doctoral thesis, H. Thunsdorff proved the following inequality.

Theorem [T]. If $f:[0,1] \to \mathbb{R}$ is a nonnegative, convex function such that $f(0) = 0$ and $0 < m \leqslant n < +\infty$, then

$$(1) \qquad \left[(m+1)\int_0^1 f^m\,dx\right]^{1/m} \leqslant \left[(n+1)\int_0^1 f^n\,dx\right]^{1/n}.$$

(See [NS] for an elementary proof of this inequality.)

It was pointed out in [N] that the classical inequality

$$(2) \qquad \left[\int_0^1 f^m\,dx\right]^{1/m} \leqslant \left[\int_0^1 f^n\,dx\right]^{1/n},$$

for nonnegative, measurable functions $f:[0,1] \to \mathbb{R}$ and $0 < m \leqslant n < +\infty$, implies the inequality

$$(3) \qquad \left[(m+1)\int_0^1 f^m\,dx\right]^{1/m} \leqslant e\left[(n+1)\int_0^1 f^n\,dx\right]^{1/n},$$

where the constant e is sharp even for the subclass of nondecreasing function $f:[0,1] \to \mathbb{R}$. In the same paper [N], a class of nondecreasing functions for which the inequality (1) holds was investigated. We give the theorem below for completeness sake.

Theorem [N]. Let $f:[0,1] \to \mathbb{R}$ be a nondecreasing function with $f(0) = 0$. If the closure of the planar set $\{(x,y) \mid f(x) \leqslant y$ and $x \in [0,1]\}$ is star-like with respect to the origin $(0,0)$ and $0 < m \leqslant n < +\infty$, then we have that the inequality (1) holds true.

F. Schnitzer and P. Schöpf [SS] extended the above theorem to the multidimensional case as follows.

Theorem [SS]. Let B be the closed unit ball in $\mathbb{R}^k$ and μ_k be Lebesgue measure on $\mathbb{R}^k$. Suppose $f: B \to \mathbb{R}$ is a nonnegative, measurable function such that the subset $A(f) = \{(x,z) \mid z \geq f(x) \text{ and } x \in B\}$ of $\mathbb{R}^{k+1}$ has the property that the segment joining $(0,0)$ to (x,z) in $\mathbb{R}^{k+1}$ is contained in $A(f)$ for each $(x,z) \in A(f)$. Then, for $0 < m \leq n < +\infty$, we have

$$\left[\frac{m+k}{k}\int_B \frac{f^m}{\mu_k(B)}\,dx\right]^{1/m} \leq \left[\frac{n+k}{k}\int_B \frac{f^n}{\mu_k(B)}\,dx\right]^{1/n}. \tag{4}$$

In the present note we will prove a moment inequality for nondecreasing functions in a measure theoretic setting. This inequality will include the classical inequality (2), the Thunsdorff inequality (1) and the Schnitzer - Schöpf inequality (4). The main theorem of our note will be free of dimensional considerations.

2. Preliminaries.

We discuss next some known facts and present the necessary definitions for the remainder of the note.

Suppose (Ω_i, μ_i) is a probability space and f_i is a nonnegative, real-valued, μ_i-measurable function ($i=1,2$). Then f_1 and f_2 are said to be equidistributed if $\mu_1(\{\omega_1 \mid f_1(\omega_1) > y\}) = \mu_2(\{\omega_2 \mid f_2(\omega_2) > y\})$ for all $y \in \mathbb{R}$. It is well-known that for any nonnegative, real-valued, μ-measurable function f on a probability space (Ω, μ) there is a nondecreasing function f_* on $[0,1]$ ($f_*(1) = +\infty$ when f is unbounded) such that f_*, with Lebesgue measure on $[0,1]$, is equidistributed with f. See the discussion on monotone adjustment in [Z], page 29. The proof uses the set $\{(t,\omega) \mid t = \mu(\{\omega \mid f(\omega) > y\})\}$. Analogously, one can prove the following.

Proposition. Let (Ω_1,μ_1) and (Ω_2,μ_2) be probability spaces and $f:\Omega_1 \times \Omega_2 \to \mathbb{R}$ be a nonnegative, $\mu_1 x \mu_2$-measurable function. Let ν be a nonatomic, Borel, probability measure on $[0,1]$. Then there is a nonnegative function f_* on $\Omega_1 \times [0,1]$ such that f_* is $\mu_1 x \nu$-measurable, f and f_* are equidistributed, and $f_*(\omega_1,r)$ is nondecreasing on $[0,1]$ for each $\omega_1 \in \Omega_1$.

The function f_* is constructed from the set

$\{(\omega_1,t,y) \in \Omega_1 \times [0,1] \times \mathbb{R} \mid \nu([t,1]) = \mu_2(\{\omega_2 \in \Omega_2 \mid f(\omega_1, \omega_2) > y\})\}$.

Definition. 2.1. Let ν be a totally finite, positive measure on $[0,1]$. An extended real-valued function w on $[0,1]$ is said to be nondecreasing with respect to ν if there is a ν-measurable set D such that $\nu(D) = \nu([0,1])$ and w is a nondecreasing, real-valued function on D. When ν is also a Borel measure, this is equivalent to w being a nondecreasing, extended real-valued function on $[0,1]$ which is ν-almost everywhere real-valued.

Definition. 2.2. Let (Ω,μ) be a probability space and let $f:\Omega \to \mathbb{R}$ be nonnegative and μ-measurable. Let w be a nonnegative, extended real-valued function which is nondecreasing with respect to a Borel, probability measure ν on $[0,1]$. We say f has a nondecreasing quotent by w with respect to ν if there is a probability space (Ω^*,μ^*) and there is a nonnegative $\mu^* x \nu$-measurable function f_* on $\Omega^* \times [0,1]$ which is equidistributed with f such that the following condition holds:

There is a μ^*-measurable set E and a ν-measurable set D with $\mu^*(E) = 1$ and $\nu(D) = 1$ such that

(*) (i) $f_*(\omega^*, r)$ is μ^*-measurable for each $r \in D$,

(ii) $f_*(\omega^*, r)/w(r)$ is nondecreasing and real-valued on D for each $\omega^* \in E$.

In Definition 2.2, let $\phi_* = f_*/w$. Then $f_* = \phi_* w$, where ϕ_* is $\mu^* x \nu$-measurable and

(i) $\phi_*(\omega^*, r)$ is μ^*-measurable for each $r \in D$,

and

(ii) $\phi_*(\omega^*, r)$ is nondecreasing on $[0,1]$ for each $\omega^* \in E$.

We conclude the section with a statement of our Main Theorem. Its proof will be given in Section 4 below.

<u>Main</u> <u>Theorem</u>. Let (Ω, μ) be a probability space and ν be a nonatomic, Borel, probability measure on $[0,1]$. Further, let w be a nonnegative, nondecreasing function on $[0,1]$ with respective to ν. Then, for $0 < m \leqslant n < +\infty$ and for a nonnegative, μ-measurable function $f: \Omega \to \mathbb{R}$ which has a nondecreasing quotient by w with respect to ν, we have

(5) $$\frac{\left[\int_\Omega f^m \, d\mu\right]^{1/m}}{\left[\int_0^1 w^m \, d\nu\right]^{1/m}} \leqslant \frac{\left[\int_\Omega f^n \, d\mu\right]^{1/n}}{\left[\int_0^1 w^n \, d\nu\right]^{1/n}},$$

provided $0 < \int_0^1 w^n \, d\nu < +\infty$.

We observe that inequality (5) reduces to the classical inequality

(6) $$\left[\int_\Omega f^m \, d\mu\right]^{1/m} \leqslant \left[\int_\Omega f^n \, d\mu\right]^{1/n}$$

for any nonnegative, μ-measurable function $f: \Omega \to \mathbb{R}$ when w is 1 and ν is Lebesgue measure. This observation is a consequence of the fact that f is equidistributed with a nondecreasing function on $[0,1)$.

3. Two Lemmas.

For the first lemma, we assume that (Ω^*,μ^*) and ν satisfy the conditions of Definition 2.2. Let f_* satisfy the condition (*). Then, for $0 < m$ and $r \in [0,1]$, let $F_m(r)$ be $\left[\int_{\Omega^*} f_*{}^m(\omega^*,r)\ d\mu^*(\omega^*)\right]^{1/m}$ when the integral exists (possibly $+\infty$), and be 0 in the contrary case.

Lemma. 3.1. Under the above assumptions, let $W_m(r) = F_m(r)/w(r)$ when $w(r) > 0$ and $W_m(r) = 0$ when $w(r) \leqslant 0$. Then W_m is nondecreasing with respect to ν. Consequently, $F_m = W_m w$ is nondecreasing with respect to ν.

Proof. Let E and D be as in Definition 2.2, and let $r_1,r_2 \in D$ with $r_1 < r_2$. Then $0 \leqslant f_*(\omega^*,r_1)/\omega(r_1) \leqslant f_*(\omega^*,r_2)/w(r_2)$ for $\omega^* \in E$. Hence, $0 \leqslant W_m(r_1) \leqslant W_m(r_2)$, and the first statement follows.

Lemma. 3.2. Let ν be a nonatomic, Borel, probability measure on $[0,1]$. Suppose g and h are nonnegative, extended real-valued, Borel measurable function on $[0,1]$ which are nondecreasing with respect to ν. Suppose further that $p \in (0,1]$ is such that
$\nu(\{r \in [0,p) \mid g(r) < h(r)\}) + \nu(\{r \in [p,1] \mid g(r) > h(r)\}) = 0$.

If $k > 1$ and $\int_0^1 g\ d\nu = \int_0^1 h\ d\nu < +\infty$, then

$$\int_0^1 g^k\ d\nu \ \leqslant\ \int_0^1 h^k\ d\nu.$$

Proof. If $\nu(\{r \in [p,1] \mid g(r) < h(r)\}) = 0$ or $\nu(\{r \in [0,p) \mid$

$g(r) > h(r)\}) = 0$, then $\int_0^1 g\, d\nu = \int_0^1 h\, d\nu < +\infty$ implies $g(r) = h(r)$ for ν-almost every $r \in [0,1]$. Hence the conclusion is true.

Next suppose $\nu(\{r \in [p,1] \mid g(r) < h(r)\}) > 0$ and $\nu(\{r \in [0,p] \mid g(r) > h(r)\}) > 0$. There is a Borel set D contained in the support of the Borel, probability measure ν such that $\nu(D) = 1$ and both g and h are nondecreasing on D. Since $\int_0^1 h\, d\nu < +\infty$, we may assume further that $h(r) < +\infty$ for all $r \in D$. Let $S = \sup\{g(r) \mid r \in D \cap [0,p)$ and $g(r) \geq h(r)\}$. Then, $0 \leq h(r) \leq g(r) \leq S$ for ν-almost all $r \in D \cap [0,p)$, and $S \leq g(r) \leq h(r) < +\infty$ for ν-almost all $r \in D \cap [p,1]$. Because $\nu(\{r \in [p,1] \mid g(r) < h(r)\}) > 0$, we have $S < +\infty$. For convenience, we may assume $g(r) = h(r) = 0$ on each of the two exceptional sets and on $[0,1]-D$. We assume $\int_0^1 h^k\, d\nu < +\infty$ because in the contrary case the conclusion of the Lemma is true. Then we infer from $S < +\infty$ that $\int_0^1 g^k\, d\nu$ is also finite. The Fubini Theorem gives $\int_0^1 h\, d\nu = \int_0^1 \int_0^{h(r)} dy\, d\nu(r)$ and $k^{-1} \int_0^1 h^k\, d\nu = \int_0^1 \int_0^{h(r)} y^{k-1}\, dy\, d\nu(r)$. The corresponding formulas hold for g, also. From the equality $\int_0^p (g-h)\, d\nu = \int_p^1 (h-g)\, d\nu \geq 0$, we get

$$\int_0^p \int_{h(r)}^{g(r)} y^{k-1} dy\, d\nu(r) \leq \int_0^p \int_{h(r)}^{g(r)} S^{k-1} dy\, d\nu(r) = S^{k-1} \int_0^p (g-h)\, d\nu$$

$$= S^{k-1} \int_p^1 (h-g)\, d\nu \leq \int_p^1 \int_{g(r)}^{h(r)} y^{k-1}\, dy\, d\nu(r).$$

Or,

$$0 \leq -\int_0^p \int_{h(r)}^{g(r)} y^{k-1}\, dy\, d\nu(r) + \int_p^1 \int_{g(r)}^{h(r)} y^{k-1}\, dy\, d\nu(r)$$

$$= \int_0^1 \int_0^{h(r)} y^{k-1}\, dy\, d\nu(r) - \int_0^1 \int_0^{g(r)} y^{k-1}\, dy\, d\nu(r)$$

$$= k^{-1} \left[\int_0^1 h^k\, d\nu - \int_0^1 g^k\, d\nu\right],$$

and the Lemma is completely proved.

For an application of Lemma 3.2, we derive the next classical inequality without the aid of the Hölder Inequality.

Corollary. 3.3. Let (Ω,μ) be a probability space and let f be a nonnegative, μ-measurable, real-valued function. Then, for $0 < m \leqslant n < +\infty$, we have that

$$\left[\int_\Omega f^m\, d\mu\right]^{1/m} \leqslant \left[\int_\Omega f^n\, d\mu\right]^{1/n}. \tag{6}$$

Proof. The inequality follows from the bounded function case. Hence we assume f is bounded. Let $f_*:[0,1] \to \mathbb{R}$ be the nondecreasing function which is equidistributed with f given by the monotone adjustment of f. Let ν be Lebesgue measure, $g(r) \equiv \left[\int_0^1 f_*^m\, dx\right]$, and $h(r) = f_*^m(r)$, $r \in [0,1]$. Using $k = n/m$, we have by Lemma 3.2 that

$$\left[\left(\int_0^1 f_*^m\, dx\right)^{1/m}\right]^n \leqslant \int_0^1 f_*^n\, dx.$$

That is $\left[\int_\Omega f^m\, d\mu\right]^{1/m} \leqslant \left[\int_\Omega f^n\, d\mu\right]^{1/n}$, and the Corollary is proved.

4. Proof of the Main Theorem.

We use the notations of Lemma 3.1. Fix $m < n$. If $\left[\int_\Omega f^n\, d\mu\right]^{1/n} = +\infty$ the inequality (5) is true. Hence, we assume that $\left[\int_\Omega f^n\, d\mu\right]^{1/n} < +\infty$. From inequality (6), we have that

$\left[\int_\Omega f^m \, d\mu\right]^{1/m} < +\infty$ and $0 < \left[\int_0^1 w^m \, d\nu\right]^{1/m} < +\infty$. Let C_m be the left-hand side of the inequality (5). Then for $g = (C_m w)^m$ and $h = (F_m)^m \equiv (W_m w)^m$ on $[0,1]$, we have $\int_0^1 g \, d\nu = \int_0^1 h \, d\nu < +\infty$. Hence, we can apply Lemma 3.2 if the appropriate p exists.

First suppose $\nu(\{r \mid g(r) < h(r)\}) = 0$. Then, $\int_0^1 g \, d\nu = \int_0^1 h \, d\nu < +\infty$ implies $g = h$ ν-almost everywhere. In this case, let $p = 1$.

Next suppose $\nu(\{r \mid g(r) < h(r)\}) > 0$. Let D be the ν-measurable set in Definition 2.2. Then $\nu(\{r \in D \mid g(r) < h(r)\}) > 0$. Moreover, we infer from Lemma 3.1 that $g(r_1) < h(r_1)$ implies $g(r_2) < h(r_2)$ when $r_1, r_2 \in D$ and $r_1 < r_2$. Let $p = \inf \{r \in D \mid g(r) < h(r)\}$. Since $\int_0^1 g \, d\nu = \int_0^1 h \, d\nu < +\infty$ and ν is a nonatomic, probability measure, we have that $p > 0$. Consequently,

$$r \in [0,p) \cap D \Rightarrow g(r) \geqslant h(r)$$

and

$$r \in [p,1] \cap D \Rightarrow g(r) \leqslant h(r).$$

Hence the appropriate p exists. Since $k = n/m > 1$, we have

$$C_m^n \int_0^1 w^n \, d\nu = \int_0^1 g^{n/m} \, d\nu \leqslant \int_0^1 h^{n/m} \, d\nu$$

$$= \int_0^1 (F_m)^n \, d\nu \leqslant \int_0^1 (F_n)^n \, d\nu$$

$$= \int_0^1 \int_{\Omega^*} f_*{}^n(\omega^*, r) \, d\mu^*(\omega) \, d\nu(r).$$

Or,

$$\frac{\left[\int_{\Omega} f^m \, d\mu\right]^{1/m}}{\left[\int_0^1 w^m \, d\nu\right]^{1/m}} \leq \frac{\left[\int_{\Omega} f^n \, d\mu\right]^{1/n}}{\left[\int_0^1 w^n \, d\nu\right]^{1/n}},$$

and the Theorem is proved.

5. Remarks.

In Section 1 we stated the Schnitzer-Schöpf Theorem. Let B be the closed unit ball in $\mathbb{R}^k$, $\partial B = \Omega^*$ be the boundary of B with the normalized (k-1)-dimensional measure μ^*, and ν be the Borel measure on [0,1] given by $d\mu = k\, r^{k-1}\, dr$, and $w(r) = r$. If $f:B \to \mathbb{R}$ is a nonnegative, Lebesgue measurable function satisfying the condition of the Theorem [SS] (i.e., has a star-like epigraph with respect to the origin), then f is equidistributed with a function $f_*:\partial B \times [0,1] \to \mathbb{R}$ for which $f_*(y,r)/w(r)$ is nondecreasing on [0,1] for each $y \in \partial B$. Theorem [SS] now follows because $\int_0^1 w^m \, d\mu = k/(m+k)$. (See the proof in [SS].)

For other references on Thunsdorff's Inequality, see [M].

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