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PLANAR SETS WHOSE COMPLEMENTS DO NOT CONTAIN A DENSE SET OF LINES

¶1. Introduction and terminology

Steinhaus proved that the set $A - B = \{a - b : a \in A, b \in B\}$ contains an interval whenever A and B are measurable subsets of the real line $\mathbb{R}$ with positive Lebesgue measure. The authors of [1] provide an alternate way of interpreting the Steinhaus theorem in terms of projections of the subset $A \times B$ of $\mathbb{R}^2$; they prove among other things that there exists a residual set in $\mathbb{R}^2$ no projection of which contains an interval. We prove the projection (measure projection) of a compact set in $\mathbb{R}^2$ to be compact (resp. an F_σ set) in $\mathbb{R}$. We employ the proof of Steinhaus' theorem using convolutions in ([3, p. 296]) to show that the measure projection of $E = A \times B$ is open in its projection whenever A and B are measurable sets with positive finite Lebesgue measure. Contrary to Theorems 2 and 3 of [1] the analogous statements are not true when A and B have either the property of Baire or, are sets of the second category. However, if the sets A and B are assumed to possess both the above properties then the category projection of E has nonempty interior; this follows from the method in [6, p. 21]. Finally, there exists a residual set E in $\mathbb{R}^2$ with full measure such that its projection has empty interior for each linear function f in a dense set. As a corollary it follows that such a set does not contain any rectangle $A \times B$ with both A and B having any one of the properties:

dense G_δ; Baire property and of the second category; positive measure.

<u>Terminology</u>: Let $f : \mathbb{R} \to \mathbb{R}$ be continuous linear, i.e. there exists an $m \in \mathbb{R}$ such that $f(x) = mx$ for each $x \in \mathbb{R}$. For $c \in \mathbb{R}$ we denote by f_c itself the graph of $f_c : \mathbb{R} \to \mathbb{R}$ defined by $f_c(x) = f(x) + c$, $x \in \mathbb{R}$. The Lebesgue measure on measurable subsets of $\mathbb{R}(\mathbb{R}^2)$ is denoted by λ_1 (resp. λ_2), and π_1 denotes the projection $(x,y) \mapsto x$ of $\mathbb{R}^2$ onto $\mathbb{R}$.

Let $E \subset \mathbb{R}^2$. Following [1] we define the <u>f-projection</u>, <u>f-measure projection</u> and <u>f-category projection</u> of E, denoted by $P(f,E)$, $Q(f,E)$ and $R(f,E)$ as below:

$$P(f,E) = \{c \in \mathbb{R} : f_c \cap E \neq \phi\},$$

$$Q(f,E) = \{c \in \mathbb{R} : \lambda_1(\pi_1(f_c \cap E)) > 0\},$$

and $$R(f,E) = \{c \in \mathbb{R} : \pi_1(f_c \cap E) \text{ is of second category in } \mathbb{R}\}.$$

In general, $Q(f,E)$ and $R(f,E)$ are subsets of $P(f,E)$, and examples are easily constructed to show that both are proper subsets even for relatively simple sets $E = A \times B$. The lines 3-6 on page 207 of [1] state that $Q(f,E)$ fills up almost all of $P(f,E)$ in the sense of measure whenever $\lambda_2(E) > 0$. Unfortunately this is false as can be seen by taking E to be the union of the sets $[1,3] \times [-1,0]$, $[1,3] \times [4,5]$, $\{(x,y) : x = 2 \text{ and } 0 \leq y \leq 4\}$, and f to be the identity function on $\mathbb{R}$. For we have $P(f,E) = [-4,4]$ whereas $Q(f,E) = (-4,-1) \cup (1,4)$ which does not fill up almost all of $P(f,E)$ in the sense of measure. However, in case E has full measure (i.e. its complement is a null set) then it follows from invariance of λ_2 under rotations and Fubini's theorem that $Q(f,E)$

does fill up almost all of P(f,E) in the sense of measure.

Let us collect some facts that follow directly from the definitions. The complement of a set A is denoted by A'.

1.1. PROPOSITION: Let A,B be subsets of $\mathbb{R}$, E be the rectangle $A \times B$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous and linear. Then

(i) $P(f,E) = B - f(A)$

(ii) $P(f,E)' = \{c : c + f(A) \subset B'\}$

and (iii) $Q(f,E) = \{c : \lambda_1(A \cap f_c^{-1}(B)) > 0\}$,

where A and B are measurable.

1.2. COROLLARY: Let A be of second category, B be residual and $E = A \times B$. Then $P(f,E) = \mathbb{R}$ for every continuous linear function $\not\equiv 0$.

PROOF: As f preserves sets of the second category, f(A) is of the second category, so is c + f(A). Since the set B' is of the first category we have $P(f,E)' = \phi$ from part (ii), i.e. $P(f,E) = \mathbb{R}$.

1.3. REMARK: The above corollary is no longer true if A and B are of the second category; an example is provided by Theorem 4 of [1]. However, see Theorem 2.6 below.

1.4. PROPOSITION: Let $E \subset \mathbb{R}^2$ and f be linear and continuous. Then $P(f,E)^\circ = \phi$ iff E' contains a dense set of lines each parallel to f.

PROOF: As in part (ii) of Proposition 1.2, we have $P(f,E) = \{c : f_c \subset E'\}$, and so $P(f,E)^\circ = \phi$ if $P(f,E)'$ is dense in $\mathbb{R}$, which is equivalent to the condition stated.

¶2. Let us first relate in Proposition 2.1 the compactness of $E \subset \mathbb{R}^2$ to the properties of the projections.

2.1. PROPOSITION: If E is compact in $\mathbb{R}^2$ then for every continuous $f : \mathbb{R} \to \mathbb{R}$

(a) $P(f,E)$ is compact in $\mathbb{R}$

and (b) $Q(f,E)$ is an F_σ set in $\mathbb{R}$.

PROOF: (a) We have $c \in P(f,E)$ iff there exists $(x,y) \in f_c \cap E$, i.e. $c = y - f(x)$. Consider the map $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined by $\phi(x,y) = y - f(x)$, $(x,y) \in \mathbb{R}^2$. Then ϕ is continuous since f is; as E is compact so is the set

$$\begin{aligned}\phi(E) &= \{y - f(x) : (x,y) \in E\} \\ &= \{c \in \mathbb{R} : \exists (x,y) \in E \cap f_c, c = y - f(x)\} \\ &= P(f,E).\end{aligned}$$

(b) With E and f being as above, let us write $P = P(f,E)$. Then P is compact by (a), Let $\alpha = \inf P$ and $\beta = \sup P$. Define $\phi : [\alpha,\beta] \to \mathbb{R}$ by $\phi(c) = \lambda_1(\pi_1(f_c \cap E))$, $c \in [\alpha,\beta]$. Then $Q \equiv Q(f,E) = \{c : \phi(c) > 0\} = \bigcup_{n=1}^{\infty} Q_n$, where $Q_n = \{c : \phi(c) \geq \frac{1}{n}\}$ for every natural number n. To prove Q to be an F_σ it clearly suffices to show each Q_n to be closed, and this will follow from the uppersemicontinuity of ϕ. To verify the latter, let $\{c_k\}$ be a sequence in $[\alpha,\beta]$ that converges to $c \in [\alpha,\beta]$. We need to show that $\overline{\lim}\ \phi(c_k) \leq \phi(c)$.

For any sequence $\{A_k\}$ of sets in a matric space X the limit superior, LsA_k is defined in [5, p. 337] and shown to be the following set:

$LsA_k = \{x \in X : \exists$ a subsequence $\{A_i\}$ of A_k and $x_i \in A_i$ for each i, such that $x_i \to x$ as $i \to \infty\}$. We claim that $Ls\pi_1(f_{c_k} \cap E) \subset \pi_1(f_c \cap E)$.

For let $x \in Ls\pi_1(f_{c_k} \cap E)$. Then there exists a subsequence $\{c_i\}$ of $\{c_k\}$ and $x_i \in \pi_i(f_{c_i} \cap E)$ for each i such that $x_i \to x$ as $i \to \infty$. Since π_1 is the first projection there further exists a $y_i \in \mathbb{R}$ such that $(x_i, y_i) \in f_{c_i} \cap E$ for every i. We have $y_i = f(x_i) + c_i$, and as f is continuous the sequence $\{y_i\}$ converges to $f(x) + c = y$ say. Then $(x,y) \in f_c$, and as the sequence $\{(x_i,y_i)\}$ is contained in the closed set E and converges to (x,y), we have $(x,y) \in E$ and so $(x,y) \in E \cap f_c$, or $x \in \pi_1(E \cap f_c)$ as claimed.

Thus we obtain $\phi(c) = \lambda_1(\pi_1(f_c \cap E)) \geq \lambda_1(Ls\pi_1(f_{c_k} \cap E))$. For any sequence $\{A_k\}$ we have [5, p. 337] $LsA_k \supset \overline{\lim}\, A_k$, and so we get

$$\begin{aligned}\phi(c) &\geq \lambda_1(\overline{\lim}\, \pi_1(f_{c_k} \cap E)) \\ &\geq \overline{\lim}\, \lambda_1(\pi_1(f_{c_k} \cap E)) \text{ by Fatou's lemma} \\ &= \overline{\lim}\, \phi(c_k),\end{aligned}$$

and so is uppersemicontinuous as asserted. This completes the proof of (b).

2.2. <u>REMARK</u>: Part (a) is false when E is not compact as we see by taking E to be the open unit disk in $\mathbb{R}^2$; nor can we replace "F_σ" in part (b) by "closed" as is clear from the example in Section 1. Moreover, let E be the set obtained by rotating the set in that example counterclockwise through 45° and f be as before. Then the function ϕ in the proof of

(b) is not continuous. However, ϕ is continuous for a measurable rectangle $E = A \times B$ of positive measure as we see in the proof of Theorem 2.3.

Although the next theorem is essentially known [3, p. 296], it provides an improvement of Theorem 1 of [1] when f is linear. We use the ideas of [3, p. 295] where the notions of P(f,E) and Q(f,E) are not considered, and indicate the modifications necessary in our context.

2.3. THEOREM: Let A and B be measurable sets in $\mathbb{R}$ with finite positive measure and let $E = A \times B$. Then Q(f,E) is an open subset of P(f,E) for every continuous linear function f not identically 0.

PROOF: First let $f(x) = -x$, $x \in \mathbb{R}$, which is the case considered in [3, p. 296]. Define $\phi : \mathbb{R} \to \mathbb{R}_+$ by the convolution of the characteristic functions χ_A and χ_B (of the sets A and B) in L^2:

(1) $\phi(c) = \chi_A * \chi_B(c) = \int \chi_A(c+y)\chi_B(-y)\,d\lambda_1(y)$, $c \in \mathbb{R}$. The integrand in (1) has the support $(-c+A) \cap (-B)$, and so $\phi(c) > 0$ iff $\lambda_1((-c+A) \cap (-B)) > 0$

iff $\lambda_1(A \cap (c-B)) > 0$ (as λ_1 is invariant under translations)

iff $\lambda_1(A \cap f_c^{-1}(B)) > 0$ (as $f^{-1}(x) = -x$ and $f_c^{-1}(x) = c - x$) and so

$\phi(c) > 0$ iff $c \in Q(f,E)$.

Since ϕ is continuous [3, p. 295] its support Q(f,E) is an open set.

In general, let $f(x) = mx$ with $m \neq 0$. We now have, from (iii) of Proposition 1.1,

$$\begin{aligned} Q(f,E) &= \{c : \lambda_1(A \cap f_c^{-1}(B)) > 0\} \\ &= \{c : \lambda_1(A \cap \tfrac{1}{m}(B-c)) > 0\} \\ &= \{c : \lambda_1(mA \cap (B-c)) > 0\}, \end{aligned}$$

for $\lambda_1(C) > 0$ iff $\lambda_1(mC) > 0$, and so we only have to replace A by mA in the special case considered first in order to complete the proof.

2.4. COROLLARY [1, Theorem 1 for linear f]. Let A and B be measurable sets in $\mathbb{R}$ with finite positive measure and f be non-zero linear continuous. Then $Q(f,A\times B)$ contains an open interval.

A set E has the property of Baire ([6, p. 19]) if $E = O \Delta A$ with O open and A first category, where Δ denotes symmetric difference. The class of sets having the property of Baire is the sigma-algebra generated by the open sets together with sets of first category ([6, p. 19]); every F_σ set and each G_δ set has the property of Baire.

2.5. REMARK: The analogues of Theorem 2.3 are false when A and B are assumed either to have the property of Baire or to be of second category. For an example of the latter, see Theorem 4 of [1]. For an example of the former, let A be the set $\mathbb{Q}$ of rational numbers and B be the set $\mathcal{J}$ of irrational numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be linear and continuous, defined by $f(x) = mx$, $x \in \mathbb{R}$, for some $m \in \mathbb{R}$. From part (i) of Proposition 1.1, we get $P(f,E) = \mathcal{J} - mQ$ and so $P(f,E) = \begin{cases} \mathcal{J} & \text{if } m = 0 \\ \mathcal{J} - Q & \text{if } m \in Q, m \neq 0. \\ \mathbb{R} & \text{if } m \notin Q \end{cases}$

Hence $P(f,E) \subset \mathcal{J}$ whenever $m \in Q$, i.e. $P(f,E)$ has empty interior. Since A is an F_σ set and B is a G_δ set, both have the property of Baire; and as B is residual, this example shows Theorem 2 and 3 of [1] to be false. Inspired by Theorem 4.8 of [6, p. 21] we do however, have the following:

2.6. <u>THEOREM</u>: Let A and B be of the second category and have the property of Baire, and $E = A \times B$. Then $R(f,E)^\circ \neq \phi$ for every linear continuous $f \neq 0$.

<u>PROOF</u>: The proof is only a modification of the one in [6, p. 21]. Let $A = G \Delta P$ and $B = H \Delta Q$, where G and H are open and P,Q are of the first category. Since A and B are of the second category, G,H are not empty and so there exist nonempty open intervals I and J such that $I \subset G$ and $J \subset H$.

Now let $m \neq 0$; then we have for every $c \in \mathbb{R}$,

$$
\begin{aligned}
&(c+mA) \cap B \\
&= (c+(mG \Delta mP)) \cap (H \Delta Q) \\
&\supset (c+(mG \sim mP)) \cap (H \sim Q) \quad \text{where } A \sim B = A \cap B' \\
&= (c+mG) \cap (c+mP)' \cap H \cap Q' \\
&= (c+mG) \cap H \cap (c+mP)' \cap Q' \\
&= (c+mG) \cap H \sim ((c+mP) \cup Q).
\end{aligned}
$$

But then we have $(c+mG) \supset (c+mI)$ and $H \supset J$, and so we obtain $(c+mA) \cap B \supset (c+mI) \cap J \sim ((c+mP) \cup Q)$. As on P. 21 of [6] it follows that there exists an open interval 0 such that for every $c \in 0$ the set $(c+mI) \cap J$ contains a nonempty interval. Hence the set $(c+mA) \cap B$ contains a nonempty open interval minus a set of the first category, i.e. for every $c \in 0$ the set $(c+mA) \cap B$ is of the second category, or the set $A \cap \frac{(B-c)}{m}$ is of the second category as well. Since the latter set is $A \cap f_c^{-1}(B)$ we have

$$0 \subset \{c : A \cap f_c^{-1}(B) \text{ is of second category}\}$$

i.e. $0 \subset R(f,E)$. Hence the theorem.

2.7. <u>REMARK</u>: Answering a question raised in [1], Roy O. Davies [2] constructed with the help of continuum hypothesis a linear set A of the second category such that $A \times A$ has full planar outer measure and $P(f,A\times A)^\circ = \phi$ for every linear f; Martin's Axiom is employed by Tomasz Katkaniec [4] to give a linear set A of the second category for which $R(f,A\times A)^\circ = \phi$ for every linear f. These sets cannot have the property of Baire by Theroem 2.6. In the same issue of the <u>Real Analysis Exchange</u> (p. 230) it is remarked that a Besicovitch Borel set $E \subset \mathbb{R}^2$ of full plane measure has the property $P(f,E)^\circ = \phi$ for each f. However, it is relatively easy to find such a set for which $P(f,E)^\circ = \phi$ for a dense set of functions:

Let us recall that the space of all continuous linear functions may be identified with $\mathbb{R}$.

2.8. <u>THEOREM</u>: There exists a residual set E in $\mathbb{R}^2$ with full measure such that P(f,E) has empty interior for a dense set of linear functions f.

<u>PROOF</u>: Let Q denote the set of rationals; we define the complement E' of E. For each $m \in \mathbb{R}$, $r \in Q$ let

$$L_{m,r} = \{(x,y) : x \in \mathbb{R},\ y = mx + r\}.$$

For each fixed m, the set $L_m = \cup\{L_{m,r} : r \in Q\}$ is of the first category in $\mathbb{R}^2$ and has measure zero. Define $E' = \cup\{L_m : m \in Q\}$. Then E' has both these properties, and so E is residual in $\mathbb{R}^2$ with full measure. For each $m \in Q$ the set E' contains a dense set of lines each parallel to $y = mx$ and so $P(f,E)^\circ = \phi$ for every $f \in Q$, by Proposition 1.4. Hence the theorem.

2.9. <u>COROLLARY</u>: There exists a dense G_δ set E with full measure in $\mathbb{R}^2$ which contains no measurable rectangle $A \times B$ with A and B satisfying any one of the following properties:

(i) dense G_δ,

(ii) positive measure,

(iii) property of Baire and of the second category.

<u>PROOF</u>: Let E be the dense G_δ set constructed in Theorem 2.8. In case E contains a rectangle $A \times B$ where A and B possess

property (i), (ii) or (iii) then we have $P(f,E)^{\circ} \neq \phi$ for every $f \not\equiv 0$, by Corollary 1.2, Theorem 2.3 and Theorem 2.6 respectively. This contradicts Theorem 2.8.

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