Real Analysis Exchange Vol. 11 (1985-86)

F.S. Cater, Department of Mathematics, Portland State University,
Portland, Oregon 97207

DERIVATIVES ON COUNTABLE DENSE SUBSETS

Throughout this paper, A, B, C and D will be mutually disjoint countable dense subsets of the real line, R. In [1], p. 72, A. Bruckner observed that there exists an everywhere differentiable function f that has a proper local maximum at each point of A and at no other points, and has a proper local minimum at each point of B and at no other points. We will prove (Lemma 1) the existence of an increasing differentiable homeomorphism of R onto R that maps certain countable subsets of R the way we please. (Compare with [4], and also note the Remark on p. 72 of [1].) We use Lemma 1, together with one prototype function from [1], to prove:

Theorem 1. There is an everywhere differentiable function F on R that has a proper local maximum at each point of A and at no other other points, has a proper local minimum at each point of B and at no other points, and is increasing at each point of C and decreasing at each point of D.

Of course F must increase (decrease) at uncountably many points, so C(D) cannot include all such points. We also use Lemma 1 to construct a variety of types of pathological functions. The first type we consider are continuous functions that have no derivative, finite or infinite, at any point.

Theorem 2. There is a continuous function F on R such that at each $x \in R$, we have $[D_+F(x), D^+F(x)] \cup [D_-F(x), D^-F(x)] = [-\infty, \infty]$, where D^+, D_+, D^-, D_- denote the usual Dini derivates, and such that

$$D_+F(x) = \infty \text{ for } x \in A, \quad D^+F(x) = -\infty \text{ for } x \in B,$$

$$D_-F(x) = \infty \text{ for } x \in C, \quad D^-F(x) = -\infty \text{ for } x \in D.$$

By a <u>knot</u> point of f, we mean an $x \in R$ at which

$$D^+f(x) = D^-f(x) = \infty, \quad D_+f(x) = D_-f(x) = -\infty.$$

We use Lemma 1 to construct a function on R, absolutely continuous on every compact interval, that has some knot points and other unusual points in every interval.

Theorem 3. There is a function F on R, absolutely continuous in every compact interval, such that

$$F'(x) = 0 \text{ for } x \in A, F'(x) = \infty \text{ for } x \in B, F'(x) = -\infty \text{ for } x \in C,$$

and each $x \in D$ is a knot point of F.

We say that a nonconstant function f on an interval I is <u>singular</u> on I if f is of bounded variation on I and $f' = 0$ almost everywhere on I. We use Lemma 1 to prove that Theorem 3 works when "singular" replaces "absolutely continuous".

Theorem 4. There is a function F on R, continuous and singular in every compact interval, such that

$$F'(x) = 0 \text{ for } x \in A, F'(x) = \infty \text{ for } x \in B, F'(x) = -\infty \text{ for } x \in C,$$

and each $x \in D$ is a knot point of F.

Lemma 1 can be used to prove the existence of certain kinds of derivatives. In [3] Katznelson and Stromberg gave a nice example of a derivative on R that is positive on A and negative on B. Of course such a derivative must be discontinuous at any point where it is nonzero. On the other hand, it might be continuous or discontinuous at any point where it is zero. We conclude with

Theorem 5. There is an everywhere differentiable function F on R such that $F'(x) > 0$ for $x \in A$, $F'(x) < 0$ for $x \in B$, F' vanishes and is discontinuous at each $x \in C$, and F' vanishes and is continuous at each $x \in D$.

If A and B are countable dense subsets of R, it is easy to construct an increasing homeomorphism of R onto R that maps A onto B. We show that we can make the homeomorphism differentiable.

Lemma 1. Let A, B, C, D be mutually disjoint countable dense subsets of R and let A_o, B_o, C_o, D_o likewise be mutually disjoint countable dense subsets of R. Then there exists a continuously differentiable function g mapping R onto R such that $g(A) = A_o$, $g(B) = B_o$, $g(C) = C_o$, $g(D) = D_o$, and $g' \geq 1$ on R.

Proof. Let (d_n) be an enumeration of D, (d_n') be an enumeration of D_o, (c_n') of C_o, (c_n) of C, (b_n') of B_o, (b_n) of B, (a_n') of A_o, (a_n) of A where $a_1 \neq 0$. We will construct a sequence of polynomials p_n and a sequence of points u_n by induction on n. Let $u_1 = a_1$ and put $p_1(x) = rx$ where $0 < r < \frac{1}{4}$ and $2u_1 + p_1(u_1) \in A_o$. This can be done because A_o is dense in R. Then $r + r|x| < 2^{-1}$ for $|x| < 1$. Suppose that $n \geq 2$ and polynomials p_j and points $u_j \in A \cup B \cup C \cup D$ $(1 \leq j \leq n - 1)$ have been chosen so that $|p_j(x)| + |p_j'(x)| < 2^{-j}$ for $|x| < j$ and $p_j'(x) > -2^{-j}$ for all x, $p_j(u_i) = 0$ for $i < j$, and

(1) for $j \equiv 1 (\mathrm{mod}\ 8)$, u_j is the element in $A\setminus\{u_1, \cdots, u_{j-1}\}$ with smallest index and $2u_j + \Sigma_{i=1}^{j} p_i(u_j) \in A_o$,

(2) for $j \equiv 2 (\mathrm{mod}\ 8)$, $u_j \in A\setminus\{u_1, \cdots, u_{j-1}\}$ and $2u_j + \Sigma_{i=1}^{j} p_i(u_j)$ is the element in $A_o \setminus \cup_{k=1}^{j-1} \{2u_k + \Sigma_{i=1}^{j-1} p_i(u_k)\}$ with smallest index,

(3) for $j \equiv 3 (\mathrm{mod}\ 8)$, same as (1) with B and B_o in place of A and A_o,

(4) for $j \equiv 4 (\mathrm{mod}\ 8)$, same as (2) with B and B_o in place of A and A_o,

(5) for $j \equiv 5 (\mathrm{mod}\ 8)$, same as (1) with C and C_o in place of A and A_o,

(6) for $j \equiv 6 (\mathrm{mod}\ 8)$, same as (2) with C and C_o in place of A and A_o,

(7) for $j \equiv 7 (\mathrm{mod}\ 8)$, same as (1) with D and D_o in place of A and A_o,

(8) for $j \equiv 0 (\mathrm{mod}\ 8)$, same as (2) with D and D_o in place of A and A_o.

For $n \equiv 1 (\mathrm{mod}\ 8)$, let $q(x) = (x - u_{n-1})^{n+1} \Pi_{j<n-1}(x - u_j)$ and let $r > 0$ be so small that $r|q(x)| + r|q'(x)| < 2^{-n}$ for $|x| < n$ and $rq'(x) > -2^n$ for all x. This can be done because $\deg q = 2n - 1$ is odd. Let u_n be the element in $A\setminus\{u_1, \cdots, u_{n-1}\}$ with smallest index, and choose s $(0 < s < r)$ such that $2u_n + \Sigma_{i=1}^{n-1} p_i(u_n) + sq(u_n) \in A_o$. This can be done because A_o is dense in R and $q(u_n) \neq 0$. Put $p_n = sq$.

For $n \equiv 2 \pmod 8$, let $q(x) = (x - u_{n-1})^{n+1}\Pi_{j<n-1}(x - u_j)$ and let r be so small that $r|q(x)| + r|q'(x)| < 2^{-n}$ for $|x| < n$ and $rq'(x) > -2^{-n}$ for all x. Let $H(x) = 2x + \Sigma_{i=1}^{n-1} p_i(x)$. It follows that $H' \geq 1$, $H(R) = R$, and say $H(w) = d$ where d is the proposed image of u_n in (2). Moreover, $w \neq u_i$ $(1 \leq i \leq n - 1)$ since $H(w) \neq H(u_i)$. From the definition of q we obtain $q(w) \neq 0$. Since A is dense in R, $H(A)$ is dense in R. Choose $a \in A\backslash\{u_1,\cdots,u_{n-1}\}$ such that $H(a) - d$ and $H(a) + rq(a) - d$ have opposite sign. (Here we use the continuity of H and of q and make a close to w.) Choose s such that $0 < s < r$ and $H(a) + sq(a) - d = 0$. Make $u_n = a$ and $p_n = sq$.

The cases $n \equiv 3$, $n \equiv 5$, $n \equiv 7 \pmod 8$ are handled similarly to the case $n \equiv 1 \pmod 8$. The cases $n \equiv 4$, $n \equiv 6$, $n \equiv 0 \pmod 8$ are handled similarly to the case $n \equiv 2 \pmod 8$. So the induction on n is complete and p_n and u_n have been chosen for all n.

The series $2x + \Sigma_{i=1}^{\infty} p_i(x)$ sums to a continuously differentiable function g on R. This follows from $|p_n(x)| + |p_n'(x)| < 2^{-n}$ for $|x| < n$. (It is well to note here that had we required $\Sigma_{i=0}^{\infty} |p_n^{(i)}(x)| < 2^{-n}$ for $|x| < n$, then g would be a real analytic function. This can be verified by using the remainder in Taylor's Theorem.) By construction, $p_n(u_i) = 0$ for $n > i$. It follows, then, that $g(A) = A_0$, $g(B) = B_0$, $g(C) = C_0$, $g(D) = D_0$. Finally,

$$g' = 2 + \Sigma_{i=1}^{\infty} p_i' \geq 2 - \Sigma_{i=1}^{\infty} 2^{-i} \geq 1 . \qquad \square$$

Before tackling the proofs of the Theorems, we make some observations about continuously differentiable functions g with positive derivative. If f has a proper local maximum (minimum) at $g(x)$, then clearly $F(t) = f(g(t))$ also has a proper local maximum (minimum) at x. Likewise, if f is increasing (decreasing) at $g(x)$, clearly F is increasing (decreasing) at x.

The last fraction in the equation

$$(*) \qquad \frac{F(x + h) - F(x)}{h} = \left[\frac{f(g(x + h)) - f(g(x))}{g(x + h) - g(x)}\right]\left[\frac{g(x + h) - g(x)}{h}\right]$$

tends to $g'(x)$ as h tends to 0, so

$$D^+F(x) = D^+f(g(x))g'(x), \quad D_+F(x) = D_+f(g(x))g'(x),$$
$$D^-F(x) = D^-f(g(x))g'(x), \quad D_-F(x) = D_-f(g(x))g'(x).$$

Thus if $[D_+f,D^+f] \cup [D_-f,D^-f] = [-\infty,\infty]$, as in Theorem 2 at every point, then $[D_+F,D^+F] \cup [D_-F,D^-F] = [-\infty,\infty]$ at every point. If f has a knot point at $g(x)$, then F has a knot point at x. If f' exists and is continuous at $g(x)$, then F' exists and is continuous at x. The same is true when "discontinuous" replaces "continuous".

Both g and g^{-1} map sets of measure 0 to sets of measure 0 [5]. Thus if f is absolutely continuous (singular) on every compact interval, then F must be absolutely continuous (singular) on every compact interval.

Proof of Theorem 1. Let f be everywhere differentiable on R such that f has proper local maximum points in every interval and has proper local minimum points in every interval. Then f is not monotone on any interval, so the sets

$$C_1 = \{x \in R: f'(x) > 0\}, \quad D_1 = \{x \in R: f'(x) < 0\}$$

are each dense in R. Moreover f is increasing at each point of C_1 and f is decreasing at each point of D_1. Let C_0 be a countable dense subset of C_1, and let D_0 be a countable dense subset of D_1. Let A_0 consist of all points where f has a proper local maximum point, and let B_0 consist of all points where f has a proper local minimum point. Then the composition function $F(x) = f(g(x))$ suffices where g is the function in Lemma 1. Note that F is everywhere differentiable by the chain rule of differentiation. □

Proof of Theorem 2. We use [2] to construct a continuous function f on R such that

$$[D_+f(x),D^+f(x)] \cup [D_-f(x),D^-f(x)] = [-\infty,\infty]$$

for each $x \in R$, and such that each of the sets

$$A_1 = \{x \in R: D_+f(x) = \infty\},$$

$$B_1 = \{x \in R: D^+f(x) = -\infty\},$$

$$C_1 = \{x \in R: D_-f(x) = \infty\},$$

$$D_1 = \{x \in R: D^-f(x) = -\infty\},$$

is dense in R. Let A_0 be a countable dense subset of A_1, B_0 of B_1, C_0 of C_1 and D_0 of D_1. Then $F(x) = f(g(x))$ suffices where g is the function in Lemma 1. Note that F inherits the desired properties from f because g' is everywhere positive. □

Proof of Theorem 3. It is known [3], [6], that there exist increasing functions h_1 and h_2 on R, absolutely continuous on every compact interval, such that $h_2' = \infty$ on a countable dense set Y and h_2' is finite on a countable dense set X, $h_1' > 0$ on $h_2(Y)$ and $h_1' = 0$ on $h_2(X)$. Then the composite function $h_3(x) = h_1(h_2(x))$ is increasing and absolutely continuous on every compact interval. Note that h_3 maps sets of measure 0 to sets of measure 0 because h_1 and h_2 do so. Moreover, $h_3' = \infty$ on Y and $h_3' = 0$ on X by the chain rule of differentiation.

Partition X into two countable dense subsets X_1 and X_2. By the argument in the preceding paragraph, there is an increasing function h_4 on R, absolutely continuous on every compact interval, such that $h_4' = 0$ on $X_1 \cup Y$ and $h_4' = \infty$ on X_2. Then $f = h_3 - h_4$ is absolutely continuous on compact intervals, and $f' = 0$ on $X_1, f' = \infty$ on Y and $f' = -\infty$ on X_2.

Each of the sets

$$E_1 = \{x \in R: D^+f(x) = \infty\},$$

$$E_2 = \{x \in R: D_+f(x) = -\infty\},$$

$$E_3 = \{x \in R: D^-f(x) = \infty\},$$

$$E_4 = \{x \in R: D_-f(x) = -\infty\},$$

is a dense G_δ-set in R, so $E_1 \cap E_2 \cap E_3 \cap E_4$ is dense. But any point in $E_1 \cap E_2 \cap E_3 \cap E_4$ is a knot point of f. Let D_0 be a countable dense subset of $E_1 \cap E_2 \cap E_3 \cap E_4$. Let $A_0 = X_1$, $B_0 = Y$, and $C_0 = X_2$. Finally, $F(x) = f(g(x))$ suffices where g is the function in Lemma 1. □

Our approach to Theorem 4 is similar. However we must use Lemma 1 twice.

Proof of Theorem 4. Let h_1 be a strictly increasing function, continuous and singular on every compact interval. By [5], h_1 cannot have a finite Dini derivate at every point of an interval I, for otherwise h_1 would be constant on I. Thus $h_1' = \infty$ on a dense subset of R. Let Y be a countable dense set of points on which $h_1' = \infty$, (and let X be a countable dense set of points on which $h_1' = \infty$) and let X be a countable dense set of points on which $h_1' = 0$. Partition X into two countable dense subsets X_1 and X_2. By Lemma 1, there is a continuously differentiable increasing function g_1 from R to R such that $g_1(X_1 \cup Y) = X$, $g_1' \geq 1$, and $g_1(X_2) = Y$. It follows that the composite function $h_2(x) = h_1(g_1(x))$ is increasing, continuous and singular on every compact interval, $h_2' = 0$ on $X_1 \cup Y$ and $h_2' = \infty$ on X_2. Put $f = h_1 - h_2$. Then f is continuous and singular on every compact interval. Moreover, $f' = 0$ on X_1, $f' = \infty$ on Y and $f' = -\infty$ on X_2. The rest of the argument is just like the proof of Theorem 3, so we leave it. □

Proof of Theorem 5. We use [3] to find a bounded derivative f' on R that takes positive and negative values in every interval. Then the set $X = \{x \in R: f'(x) = 0\}$ is dense in R. Fix $x_1 \in X$. Construct a sequence of intervals $(a_n,b_n)_{n=1}^{\infty}$ such that $\lim a_n = x_1$, and for each n, $a_n \in X$, $b_n \in X$,

$$x_1 < a_{n+1} < b_{n+1} < a_n \quad \text{and} \quad b_n - a_n < 2^{-n}(a_n - x_1).$$

For each n let r_n be the positive number such that $\sup r_n|f'(a_n,b_n)| = 1$. Let h_1 be the function that coincides with $r_n f'$ on (a_n,b_n) for each n, and vanishes elsewhere. For $a_{n-1} > t \geq a_n$ we have

$$|\int_{x_1}^{t} h_1| \leq \int_{x_1}^{t} |h_1| \leq \Sigma_{j=n}^{\infty} \int_{a_j}^{b_j} |h_1| \leq \Sigma_{j=n}^{\infty} (b_j - a_j)$$

$$\leq \Sigma_{j=n}^{\infty} 2^{-j}(a_j - x_1) \leq (t - x_1) \Sigma_{j=n}^{\infty} 2^{-j} = 2^{1-n}(t - x_1) .$$

It follows that $\lim_{t \to x_1} (t - x_1)^{-1} \int_{x_1}^{t} h_1 = 0$, and h_1 is the derivative of its primitive function at x_1, and at all points of R as well. Note that f' and h_1 have opposite sign at no point, h_1 vanishes where f' vanishes, and h_1 is discontinuous at x_1 by construction.

Now let $\{x_n\}_{n=1}^{\infty}$ be a countable dense subset of X. For each n, we use the argument in the preceding paragraph to construct a derivative h_n such that $\sup|h_n| = 1$, h_n is discontinuous at x_n, and no two of the functions $f', h_1, h_2, \cdots, h_n, \cdots$ have opposite sign at any point.

Then $f' + \Sigma_{n=1}^{\infty} 2^{-n}h_n$ is also a derivative, say of the function H, and H' is discontinuous at each x_n. Moreover, H' vanishes on X, so H' is continuous on a dense subset of X. Also H' is positive (negative) at any point where f' is positive (negative). Let A_0 be a countable dense set of points on which H' is negative, let $C_0 = \{x_n\}_{n=1}^{\infty}$ on which H' vanishes and is discontinuous, and let D_0 be a countable dense set of points on which H' vanishes and is continuous. Then $F(x) = H(g(x))$ suffices where g is the function in Lemma 1. □

Lemma 1 can also be used to construct other pathological functions. For example, there is a continuous function F on R that has an upward cusp at each point of A and at no other points, and a downward cusp at each point of B and at no other points, and each point of C is a knot point of F. To construct F, note that it is easy to construct a continuous function with upward and downward cusps in every interval; such a function must have knot points in every interval, etc. A possible topic for further study would be the possibilities when some of the sets A, B, C, D are uncountable. This is suggested in [6], for example. We used 4 pairs of sets in Lemma 1, but it clearly can be generalized to n pairs of sets. It can be proved for countably infinitely many pairs. If it is desired that g' be positive and bounded for any reason, consider g^{-1}.

REFERENCES

1. A. Bruckner, Some new simple proofs of old difficult theorems, Real Analysis Exchange, vol. 9 (1984) pp. 63-78.

2. K.M. Garg, On a residual set of continuous functions, Czechoslovak Mathematical Journal, vol. 20 #4 (1970) pp. 537-543.

3. Y. Katznelson & K. Stromberg, Everywhere differentiable nowhere monotone functions, Amer. Math. Monthly 81 (1974) pp. 349-354.

4. J.W. Nienhuys & J.G.F. Thiemann, On the existence of entire functions mapping countable dense sets onto each other, Indagationes Mathematicae, vol. 38 #4 (1976) pp. 331-334.

5. S. Saks, Theory of the Integral, Second Revised Edition, Dover, 1964, p. 271.

6. Z. Zahorski, Sur la premiere derivee, Transactions Amer. Math. Society, vol. 69 (1950) pp. 1-54, Theorem 8.