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NONABSOLUTELY CONVERGENT INTEGRALS

Interesting generalizations of the descriptive definition for nonabsolutely convergent integrals were given by H.W. Ellis [3], J. Foran [6], and C.M. Lee [11]. The most remarkable one is that of Foran, which is a classical generalization of the Denjoy integral in the wide sense, i.e., Foran's class of primitives is a class of continuous functions which contains strictly the ACG functions. The classes of primitives for the integrals of Ellis and the integrals of Lee are not classes of continuous functions and restricted to the class of continuous functions one obtains at most the class of ACG functions.

In this paper we give various extensions for each of these integrals. The classes of primitives for our generalizations are not classes of continuous functions. However, if one restricts these primitives to the continuous functions, some of these classes contain strictly the primitives in the Foran sense.

The uniqueness of the integration for the Foran integral follows by a corollary of Theorem 7.7 of [12] (p. 285).

To assure the uniqueness of our integrations we give some monotonicity theorems among which Theorem 3 is the most important. Theorem 3 generalizes Theorem 7.7 of [12] (p. 285) and its corollaries are both intrinsically interesting and useful.

For convenience if P is a well-defined property for functions defined on a certain domain, we will also use P to denote the class of all functions having the property P. The conditions (N), T_2, VB_*, VBG_*, VB, VBG, AC_*, ACG_*, AC, ACG are defined in [12]. In [6] Foran introduced conditions A(N) and B(N) and in [5] V. Ene has introduced condition E(N). If in the definition of A(N) the intervals I_k are allowed to overlap, a more restrictive condition results which we call condition $A^*(N)$.

We denote by $\mathcal{F}$ (respectively $\mathcal{F}^*$, $\mathcal{B}$, $\mathcal{E}$) the class of all continuous functions F defined on a closed interval I for which there exist a sequence $\{E_n\}$ of sets and a sequence $\{N_n\}$ of natural numbers such that $I = \cup E_n$ and F is $A(N_n)$ (respectively $A^*(N_n)$, $B(N_n)$, $E(N_n)$) on

E_n. If the functions are not supposed to be continuous, we define analogously the class $/\mathcal{F}/$ (respectively $/\mathcal{F}^{\bullet}/$, $/\mathcal{B}/$, $/\mathcal{E}/$, /ACG/). If in addition each E_n is closed and $F|E_n$ is continuous, we define in the same way the class $[\mathcal{F}]$ (respectively $[\mathcal{F}^{\bullet}]$, $[\mathcal{B}]$, $[\mathcal{E}]$, [ACG]).

We denote by $b/\mathcal{F}/$ the class of all functions $F \in /\mathcal{F}/$ such that F is bounded on each E_n. Condition [CG] is defined in [3], condition $\bar{N}$ in [9], condition (M) in [8], condition CM in [11], conditions C_n and AC_n in [1], and conditions DB_1, DB_1T_2 in [2]. If the condition Baire 1 (B_1) is replaced by O'Malley's condition Baire* 1 (A function F defined on [0,1] is B_1^* if every closed set has a portion on which the restriction of F is continuous.), we obtain conditions DB_1^* and $DB_1^*T_2$. Let $\mathcal{A}_1 \oplus \mathcal{A}_2$ denote the linear space generated by the classes of functions $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{D}$ (respectively $\mathcal{D}_{ap}$) be an additive class of functions, differentiable in a sense which is compatible with the ordinary derivative (respectively approximate derivative). We denote this derivative of F by DF (respectively $D_{ap}F$).

__Definition__. A function F is said to satisfy condition [M] (respectively $[M_*]$) on a closed set E if F is AC (respectively AC_*) on each closed subset of E on which it is continuous and VB (respectively VB_*).

__Remark 1__. a) $/\mathcal{B}/ \subset T_2$. This follows by [6] ((iv), p. 360) and [7] (Corollary 2, p. 35).

b) (N) $\subset$ [M]. This follows by the Banach-Zarecki theorem ([12], p. 227).

c) If $F \in$ ACG and $G \in$ [M], then $F+G \in$ [M]. (See the Banach-Zarecki theorem.)

d) Conditions Baire* 1 and [CG] are equivalent on a closed set E. This follows by the proof of Theorem 9.1 ([12], p. 234).

e) /ACG/ $\cap$ Baire* 1 = [ACG] on a closed set E, and if $F \in$ [ACG] and $G \in [M] \cap [CG]$, then $F+G \in [M] \cap [CG]$. This follows by the Banach-Zarecki theorem and our Remark 1,d).

f) If $F \in ACG_*$ and $G \in [M_*]$, then $F+G \in [M_*]$.

g) If $F \in \mathcal{F}$, $G \in$ AC and G is strictly increasing, Then $F \circ G \in \mathcal{F}$.

h) $[M] \subset [M_*]$. This follows by Theorem 8.8, p. 233 of [12].

i) There exists a continuous function $F \in \mathcal{B} \cap (M)$ such that $F \notin (N)$. (See [4].) There exists a continuous function $G \in (M)$ such that $G \notin T_2$. (See [8], p. 84.)

j) $/\mathcal{F}/ \subset /\mathcal{E}/$. (See [6], (iii), p. 360). If $F \in /\mathcal{F}/$ and $G \in /\mathcal{E}/$, then $F+G \in /\mathcal{E}/$.

k) $b/\mathcal{F}/ \subset /\mathcal{B}/$. (See the proof of (v), p. 362 of [6].)

Lemma 1. a) *If* $F \in /\mathcal{F}^{*}/$ *and* $G \in N$, *then* $F+G + \overline{N}$. *Moreover, if both* F *and* G *are bounded, then* $F{\cdot}G \in N$.

b) *There exist continuous functions* $F \in \mathcal{F}$ *and* $G \in \overline{N}$ *such that* $F+G \in (M)$.

c) $\mathcal{F}^{*}$ *and* $/\mathcal{F}^{*}/$ *are additive classes of functions.*

Proof. a) The proof is analogous to that of Theorem 2, p. 33 of [9].

b) Let F and G be the two continuous functions defined in [4]. Then $F+G = \phi$ (ϕ is the Cantor ternary function). F satisfies $A^{*}(2)$ on C (C is the Cantor ternary set) and $G \in (N)$ on $[0,1]$. By [9] (Lemma 1) there exists a function H strictly monotone on $[0,1]$ such that H and H^{-1} are AC and $\Lambda(B(G{\cdot}H^{-1},)H(C)) = 0$. ($\Lambda$ is linear measure and $B(F,S)$ is the graph of F restricted to S.) Then $G{\cdot}H^{-1}$ is linear on each interval contiguous to $H(C)$ and $G{\cdot}H^{-1}$ satisfies $\overline{N}$ on $[0,1]$. By Remark 1, g) $F{\cdot}H^{-1} \in \mathcal{F}$ on $[0,1]$. But on each interval contiguous to $H(C)$, $\phi{\cdot}H^{-1}$ is constant and $\phi{\cdot}H^{-1}(H(C)) = \phi(C) = [0,1]$. Therefore $\phi{\cdot}H^{-1}$ does not satisfy (M).

c) The proof is analogous to that of (vi), p. 361 of [6].

Theorem 1. a) *In the sequence of classes* $/\mathcal{F}^{*}/$, $/\mathcal{F}/$, $/\mathcal{E}/$, N, (N), $[M]$ *each class is strictly contained in all those following it.*

b) $ACG \not\subset \mathcal{F}^{*}$ *and* $\mathcal{F}^{*} \not\subset ACG$.

c) $\mathcal{E} \cap \mathcal{B} \neq \emptyset$, $\mathcal{B} \not\subset \mathcal{E}$, $\mathcal{E} \not\subset \mathcal{B}$.

d) *There exists a continuous function* F *defined on a closed interval* I, $F \in \mathcal{F}$ *and a continuous function* $G \in N$ *with* $F' = G'$ a.e. *such that* $F-G$ *is not constant on* I.

Proof. a) It is clear that each class is contained in all those following it. It remains to show that these inclusions are strict. That $/\mathcal{F}^{*}/$ is contained strictly in $/\mathcal{F}/$ follows by Lemma 1, a), b). That $/\mathcal{F}/$ is strictly contained in $/\mathcal{E}/$ follows by Theorem 5, b) of [5]. By Remark 1, j) and Lemma 1, b) it follows that $/\mathcal{E}/$ is strictly contained in $\overline{N}$. That $\overline{N}$ is strictly contained in (N) follows by [9] (p. 35). That (N) is strictly contained in [M] follows by [8] (p. 84).

b) By [9] (p. 35-36) if $F \in AC$ and $G \in \overline{N}$, then $F+G$ is not necessarily in (N). Now by Lemma 1, a) $ACG \not\subset \mathcal{F}^*$. Clearly if $F \in ACG$ and $G \in (M)$, then $F+G \in (M)$. By the proof of Lemma 1, b) we have that if $F \in \mathcal{F}^*$ and $G \in (N)$, then $F+G$ is not necessarily in (M). Therefore $\mathcal{F}^* \not\subset ACG$.

c) Since $\mathcal{F} \subset \mathcal{E} \cap \mathcal{B}$, $\mathcal{E} \cap \mathcal{B} \neq \emptyset$. Since $\phi \in \mathcal{B}$ and $\phi \notin \mathcal{E}$, $\mathcal{B} \not\subset \mathcal{E}$. ($\phi$ is the Cantor ternary function.) We show now that $\mathcal{E} \not\subset \mathcal{B}$. Let $K = \{x : x = \Sigma\, c_k/(2k+1)^k$ where $c_k = 0,2,\ldots,2k$ only$\}$ and let $a = \inf K$, $b = \sup K$. Let $\{j_k\}_k$ be a strictly increasing sequence of nonnegative integers with $j_o = 0$. For each $x \in K$ we define

$$F(x) = \sum_{k=1}^{\infty} c_{j_k}(x)/(2j_k+1)^{j_{k-1}}.$$

Extending F linearly on each interval contiguous to K we have F defined and continuous on the interval $[a,b]$. Clearly the graph on the set K can be covered by $(j_k)!$ rectangles

$$\left[\sum_{i=1}^{j_k} \frac{c_i}{(2i+1)^i}, \sum_{i=1}^{j_k} \frac{c_i}{(2i+1)^i}, + \sum_{i=j_k+1}^{\infty} \frac{2i}{(2i+1)^i}\right] \times$$

$$\left[\sum_{i=1}^{k} \frac{c_{j_i}}{(2j_i+1)^{j_{i-1}}}, \sum_{i=1}^{k} \frac{c_{j_i}}{(2j_i+1)^{j_{i-1}}} + \sum_{i=k+1}^{\infty} \frac{c_{j_i}}{(2j_i+1)^{j_{i-1}}}\right]$$

and

$$\sum_{i=j_k+1}^{\infty} 2i/(2i+1)^i < (3/2)/(2j_k+3)^{j_k};$$

$$\sum_{i=k+1}^{\infty} 2j_i/(2j_i+1)^{j_{i-1}} < (3/2)/(2j_{k+1}+1)^{j_{k-1}}.$$

Now it follows easily that F is $E(1)$ on K and $F \in \mathcal{E}$.

In what follows we show that F is B(N) on no portion of K for no natural number N. Let k be a natural number. Then for each set

$$E = K \cap \left[\sum_{i=1}^{j_k} \frac{c_i}{(2i+1)^i}, \sum_{i=1}^{j_k} \frac{c_i}{(2i+1)^i} + \sum_{i=j_k+1}^{\infty} \frac{2i}{(2i+1)^i} \right], \quad c_i =$$

$= 0,2,\ldots,2i$ the intervals $I_{c_{j_k+1},\ldots,c_{j_{k+1}-1}} =$

$$\left[\sum_{i=1}^{j_{k+1}-1} \frac{c_i}{(2i+1)^i}, \sum_{i=1}^{j_{k+1}-1} \frac{c_i}{(2i+1)^i} + \sum_{i=j_{k+1}}^{\infty} \frac{2i}{(2i+1)^i} \right]$$

are nonoverlapping and have points in common with E. If $j_{k+1} = 3j_k + 1$, then we have

$$(j_k+1)\cdot(j_k+2)\cdot\ldots\cdot(j_{k+1}-1) > \left[(j_k+1)(j_{k+1}-1)\right]^{j_k}$$

such intervals. If we cover the set $F(K \cap I_{c_{j_k+1},\ldots,c_{j_{k+1}-1}})$ with j_{k+1} intervals, then at least one of these intervals has length greater than $1/(2j_{k+1}+1)^{j_k}$. Since

$$\left[\frac{(j_k+1)\cdot(j_{k+1}-1)}{2j_{k+1}+1}\right]^{j_k} \longrightarrow \infty, \quad \text{as} \quad k \longrightarrow \infty, \quad F \notin \mathcal{B}.$$

d) the functions $F \circ H^{-1}$ and $G \circ H^{-1}$ given in the proof of Lemma 1, b) have the desired properties.

<u>**Lemma 2.**</u> <u>Suppose that</u> F <u>is continuous and satisfies condition</u> (M) <u>on</u> [a,b]. <u>Let</u> $P = \{x : +\infty > F'_{ap}(x) > 0\}$; $N = \{x : 0 > F'_{ap}(x) > -\infty\}$. <u>Then</u> $P \cup N$ <u>is nondenumerable</u>. <u>If</u> $F(a) < F(b)$, <u>then</u> $|F(P)| > F(b) - F(a)$. <u>If</u> $F(a) > F(b)$, <u>then</u> $|F(N)| > F(a) - F(b)$.

Proof. Suppose that $F(a) < F(b)$. For each $y \in [F(a),F(b)]$, let $x_y = \min\{x \in [a,b] : F(x) = y\}$. Let $E = Cl\{x_y : y \in [F(a),F(b)]\}$ ($Cl\ A$ is the closure of A). Clearly F is nondecreasing on E and $F(E) = [F(a),F(b)]$. Since F satisfies (M), $F|_E$ is AC, $|E| > 0$ and $F'_{ap}(x) \geq 0$ a.e. on E. Hence $|F(P)| \geq |F(E)| = F(b) - F(a)$.

Lemma 3. *If a function F is continuous and satisfies condition (M) on [0,1] and if $F'_{ap}(x) \geq 0$ at almost every point x where $F'_{ap}(x)$ exists, then F is montone nondecreasing on [0,1].*

Proof. Suppose that there exist $a,b \in [0,1]$, $a < b$, such that $F(b) < F(a)$. Let N be the set defined in Lemma 2. Then $N = N_0 \cup N_-$, where $N_0 = \{x : F'_{ap}(x) = 0\}$ and $N_- = \{x : 0 > F'_{ap}(x) \geq -\infty\}$. But $|F(N_0)| = 0$ and $|N_-| = 0$. By Theorem 10.8, p. 257 of [12] F is VBG on N_-. Since F satisfies (M), F is ACG on N_- and $|F(N_-)| = 0$. By Lemma 2 $|F(N)| = 0 \geq F(a) - F(b)$.

Theorem 2. *Let F be a function belonging to CM and B_1^* on a closed interval [a,b] which satisfies [M]. If $F'_{ap}(x) \geq 0$ a.e. where F'_{ap} exists, then F is nondecreasing on [a,b].*

Proof. Let $U(F) = \text{int}\{x : F \text{ is continuous at } x\}$ (int A is the interior of the set A). Suppose that $U(F) \neq (a,b)$. Let $\{I_n\}$ be the components of the open set U(F). Since F is CM, by Lemma 3 $F|_{\bar{I}_n}$ is monotone nondecreasing and $P = [a,b] - U(F)$ is perfect set. Since F is B_1^*, there exists an interval I such that $F|_{P\cap I}$ is continuous. Hence $F|_I$ is continuous and by Lemma 3 F is monotone nondecreasing on I. But this is impossible since I contains points of P.

Lemma 4. *Let F be a function belonging to D on [0,1] and let P be a closed subset of [0,1] such that F is VB_* on P. Then F is continuous on P. (D is the class of all Darboux continuous functions.)*

Proof. Let $a = \inf P$, $b = \sup P$ and let (a_k,b_k) be the intervals contiguous to P. Let $M_k = \sup F([a_k,b_k])$ and $m_k = \inf F([a_k,b_k])$. Let $c_k,d_k \in R$ be such that $a_k < c_k < d_k < b_k$. Let

$$F_1(x) = \begin{cases} F(x) & , x \in P \\ \dfrac{M_k - F(a_k)}{c_k - a_k} \cdot (x-a_k) + F(a_k) & , x \in [a_k, c_k] \\ \dfrac{m_k - M_k}{d_k - c_k} \cdot (x-c_k) + M_k & , x \in (c_k, d_k) \\ \dfrac{F(b_k) - m_k}{b_k - d_k} \cdot (x-d_k) + m_k & , x \in [d_k, b_k] \end{cases}$$

Since a function in D can have no jump discontinuities while a function in VB_* can have only jump discontinuities, F_1 is continuous on [a,b]. Hence F is continuous on P.

__Lemma 5.__ (_A generalization of Theorem 6.9_, p. 281 _of_ [12].) _Let_ F _be_ DB_1T_2 _on_ [a,b] _and let_ g _be a finite summable function. Suppose further that_ $F'(x) \leq g(x)$ _at each point_ x _at which the derivative_ $F'(x)$ _exists, except perhaps those of an enumerable set or, more generally, those of a set_ E _such that_ $|F(E)| = 0$. _Then the function_ F _is_ VB _and we have_

$$F(b) - F(a) \leq \int_a^b F'(x)dx.$$

__Proof.__ The proof is identical to that of [12] except that instead of Theorem 6.6 ([12], p. 280), we apply Theorem 2.2, p. 178 of [2].

__Theorem 3.__ (_A generalization of Theorem 7.7_, p. 285 _of_ [12]). _In order that a_ DB_1T_2 _function_ F _be_ AC _on an interval_ I_0 _it is necessary and sufficient that the function_ F _satisfy condition_ [M] _and the condition_

$$\int_P F'(x)dx < +\infty$$

where $P = \{x : 0 \leq F'(x) < +\infty\}$.

Proof. By Theorem 6.1, p. 225 of [12] and Remark 1,b)h) AC implies condition $[M_*]$. By Theorem IV, p. 473 of [3] AC implies condition T_2. Now the necessity is evident. Conversely suppose that $F \in DB_1T_2 \cap [M_*]$. Let

$$g(x) = \begin{cases} F'(x) & , x \in P \\ 0 & , x \in I_0 - P. \end{cases}$$

If $E = \{x : F'(x) = +\infty\}$, we shall have $F'(x) \leq g(x)$ at every point $x \in I_0 - E$ at which the derivative $F'(x)$ exists. By Theorem 10.1, p. 234 of [12] F is VBG_* on E. Let $E = \cup E_n$ such that $F|E_n$ is VB_*. Then F is VB_* on $\bar{E}_n$. (See [12], Theorem 7.1, p. 229.) Now Lemma 4 and condition $[M_*]$ imply that F is AC_* on E_n. Hence F is ACG_* on E. But $|E| = 0$. (See [12], p. 236.) and so $|F(E)| = 0$. By Lemma 5 F is BV on I_0. Since conditions VB and VB_* are equivalent on an interval (See [12], p. 228.) and since $F \in [M_*]$, F is AC on I_0.

Corollary 1. Every DB_1T_2 function F on $[a,b]$ which satisfies the condition $[M_*]$ and whose derivative is nonnegative at almost every point where F is derivable is montone, nondecreasing and continuous on I_0.

Remark 2. Corollary 1 can also be obtained by Theorem 2, p. 63 of [10] and the fact that F is ACG_* on $E = \{x : -\infty \leq F'(x) < 0\}$. (This fact can be verified using a proof similar to that of our Theorem 3.)

Corollary 2. ([10], Theorem 2, p. 61.) Let F be a real-valued function having the following properties on an interval: i) F is DB_1; ii) F fulfills Lusin's condition (N); iii) $F'(x) \geq 0$ at almost every point x at which F is derivable. Then F is monotone nondecreasing and continuous on the interval.

Proof. The assertion follows from Corollary 1, Remark 1,b),h) and Theorem IV, p. 473 of [3].

Corollary 3. If $F \in CM \cap B_1^* \cap [M_*]$ on $[a,b]$ and $F'(x) \geq 0$ a.e. where F is derivable, then F is monotone nondecreasing on $[a,b]$.

Proof. Using Corollary 1, the result follows by an argument analogous to the proof of Theorem 2.

Corollary 4. *If* F *and* G *are two real-valued functions defined on* $[0,1]$, $FD \in DB_1 \cap /\mathcal{B}/ \cap [M] \cap \mathcal{D}$, $G \in ACG$ *and* $DF = G'$ a.e. *on* E, $E = \{x : G'(x)$ *is finite*$\}$, *then* $G-F$ *is constant on* $[0,1]$.

Proof. Let $K(x) = G(x) - F(x)$. Clearly $K \in DB_1 \cap /\mathcal{B}/ \cap [M]$. Since $G \in ACG$, there exists a sequence $\{I_n\}$ of intervals whose union is dense in $[0,1]$ and on each of which G is AC. Hence G is derivable a.e. on I_n. Clearly $DF = G'$ a.e. on I_n. It follows that $DK = 0$ a.e. on I_n. By Corollary I, K is constant on I_n and since $K \in D$, K is constant on $\bar{I}_n$. The intervals I_n can be chosen to be maximal open intervals of constancy of K. We show that there exists only one such maximal interval, namely the interior of $[0,1]$. Suppose there is more than one such maximal interval and let $P = [0,1] - \cup I_n$. Clearly the set P is perfect. Now K is in Baire class one. Hence there exists a dense G_δ - subset $H \subset P$ on which $K|_P$ is continuous. Clearly $K \in /\mathcal{B}/$ on H. Write $H = \cup H_n$ where F is $B(N_n)$ on H_n for each n. Since H is a residual subset of the complete metric space P, there exists an interval J_1 containing points of H, together with a positive integer p such that H_p is dense in $H \cap J_1$. Now since $K|_P$ is continuous on H, $K(I \cap H_p) \supset K(I \cap H)$ for each interval $I \subset J_1$. On p. 182 of [2] it is shown that $K(H \cap I) \supset K(P \cap I)$ for each interval $I \subset J_1$. Hence K is $B(N_p)$ on $J_1 \cap P$. Since K is constant on each I_n and since K has the Darboux property on $[0,1]$, K is VB on J_1. Because K satisfies condition $[M]$, K is AC on J_1. Hence K is constant on J_1. Contradiction.

Theorem 4. *Any linear subclass* S *of* $/\mathcal{E}/ \cap DB_1 \cap \mathcal{D}$ *on* $[0,1]$ *can be taken as a class of primitives with the following properties:* a) *order*, i.e., *if* $F,G \in S$ *and* $DF \geq DG$ a.e., *then* $F(1) - F(0) \geq G(1) - G(0)$; b) *compatibility with the* $\mathcal{F}\mathcal{D}$ *primitives*, i.e., *if* $F \in S$, $G \in \mathcal{F} \cap \mathcal{D}$ *and* $DF = DG$ a.e. *on* $[0,1]$, *then* $F-G$ *is constant on* $[0,1]$.

Proof. The statement follows from Theorem 1,a), Corollary 2 and Remark 1,k).

Theorem 5. *Any linear subclass* S *of* $DB_1 \cap \bar{N} \cap \mathcal{D}$ *on* [0,1] *can be taken as a class of primitives with the following properties:* a) *order;* b) *compatibility with the* $\mathcal{F}^* \cap \mathcal{D}$ *primitives.*

Proof. The theorem follows from Lemma 1, a), Theorem 1, a), and Corollary 2.

Theorem 6. *Any linear subclass* S *of* $DB_1^* \cap [M] \cap \mathcal{D}_{ap}$ *on* [0,1] *can be taken as a class of primitives with the following properties:* a) *order;* b) *compatibility with the* ACG *primitives.*

Proof. Since the Darboux property implies condition CM ([11], p. 69), the assertion follows from Theorem 2.

Theorem 7. *Any linear subclass* S *of* $DB_1 \cap [M_*] \cap /\mathcal{B}/ \cap \mathcal{D}$ *on* [0,1] *can be taken as a class of primitives with the following properties:* a) *order;* b) *compatibility with the* ACG *primitives.*

Proof. The proof is accomplished by Corollary 1 and Remark 1, a), f).

Theorem 8. *Any linear subclass* S *of* $DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}_{ap}$ *on* [0,1] *can be taken as a class of primitives with the following properties:* a) *order;* b) *compatibility with the* ACG *primitives.*

Proof. Use Corollary 1 and Remark 1, a), c).

Theorem 9. *Any linear subclass* S *of* $DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}$ *on* [0,1] *can be taken as a class of primitives with the following properties:* a) *order;* b) *if* $F \in S$, $G \in ACG$ *and* $DF = G'$ a.e. *on* E, $E = \{x : G'(x)$ *is finite*$\}$, *then* F–G *is constant on* [0,1].

Proof. The assertion follows from Corollary 1, Remark 1, a), c) and Corollary 4.

Theorem 10. *Any linear subclass* S *of* $CM \cap B_1^* \cap [M] \cap \mathcal{D}_{ap}$ *on* [0,1] *can be taken as a class of primitives with the order property.*

Proof. The Theorem follows from Theorem 2.

Theorem 11. Any linear subclass S of $CM \cap B_1^* \cap [M] \cap T_2 \cap \mathcal{D}$ can be taken as a class of primitives with the order property on $[0,1]$.

Proof. Apply Corollary 3.

In all these cases the S - integral of DF (respectively $D_{ap}F$) on $[0,1]$ is defined to be $F(1) - F(0)$.

Remark 3. a) Theorems 5,6,7,8,9 are generalizations of Theorem XII of [3].

b) Theorems 10 and 11 are generalizations of Lee's LDG - integrals ([11], p. 72).

Examples and Remarks. By Theorem 1,a) it follows that $/\mathcal{E}/ \cap DB_1 \mathcal{D} \subset DB_1 \cap \overline{N} \cap \mathcal{D}$, by Remark 1,b) we have $DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}_{ap} \subset DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D} \subset DB_1 \cap [M_*] \cap /\mathcal{B}/ \cap \mathcal{D}$ and by [11] (p. 69) $DB_1^* \cap [M] \cap \mathcal{D}_{ap} \subset CM \cap B_1^* \cap [M] \cap \mathcal{D}_{ap}$.

Let $F_1, F_2 \in \mathcal{E} - \mathcal{F}$ with $F_1 + F_2 = \Phi$. (See [5].)

Let $H_1 \in N - \mathcal{F}^\bullet$ such that H_1 is a.e. differentiable on $[0,1]$. (See the proof of Lemma 1,b).)

Let $H_2 \in (M) - ACG$ such that H_2 is approximately differentiable a.e. on $[0,1]$. (See [4], Remark 3.)

Let $H_3 \in DB_1 \cap [M_*] \cap /\mathcal{B}/ \cap \mathcal{D}$ such that $H_3 \notin ACG_*$. (See [4], Remark 3.)

Let $H_4 \in DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}_{ap}$ such that $H_4 \notin ACG$. (See [4], Remark 3.)

Let $H_5 \in DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}$ such that $H_5 \notin ACG$. (See [4], Remark 3.)

Let $\{G_n\}$ be a strictly increasing, unbounded sequence of continuous functions, $G_n \in (N)$, $G_n(0) = 0$ and let G be a continuous function, $G \in \mathcal{F}^\bullet$ such that $G_n'(x) = G'(x)$ a.e. on $[0,1]$. (See [5].)

a) $C_n \cap /\mathcal{F}/ \cap \mathcal{D}$, $AC_n \cap /\mathcal{F}/ \cap \mathcal{D}$, $(C_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_1$, $(C_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_2$, $(AC_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_1$ and $(AC_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_2$ are linear subclasses of $/\mathcal{E}/ \cap DB_1 \cap \mathcal{D}$. Moreover, the $(C_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_1$ and $(C_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_2$ - integrals are not compatible. Also the $(AC_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_1$ and $(AC_n \cap /\mathcal{F}/ \cap \mathcal{D}) \oplus F_2$ - integrals are not compatible with each other.

b) $(C_n \cap /\mathcal{F}^{\bullet}/ \cap \mathcal{D}) \oplus H_1$ and $(AC_n \cap /\mathcal{F}^{\bullet}/ \cap \mathcal{D}) \oplus H_1$ are linear subclases of $DB_1 \cap \bar{N} \cap \mathcal{D}$.

c) $C_n \cap [\mathcal{F}] \cap \mathcal{D}_{ap}$, $AC_n \cap [\mathcal{F}] \cap \mathcal{D}_{ap}$, $[ACG] \oplus H_2$, $ACG \oplus G$ and $ACG \oplus G_n$ are linear subclasses of $DB_1^* \cap [M] \cap \mathcal{D}_{ap}$. Moreover, the $ACG \oplus G$ and $ACG \oplus G_n$ - integrals are not compatible with each other.

d) $C_n \cap b/\mathcal{F}/ \cap \mathcal{D}_{ap}$, $AC_n \cap b/\mathcal{F}/ \cap \mathcal{D}_{ap}$, $ACG \oplus H_4$, $ACG \oplus G$ and $ACG \oplus G_n$ are linear subclasses of $DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}_{ap}$.

e) $C_n \cap b/\mathcal{F}/ \cap \mathcal{D}$, $AC_n \cap b/\mathcal{F}/ \cap \mathcal{D}$ and $(ACG \cap \Delta_{ae}) \oplus H_5$ are linear subclasses of $DB_1 \cap [M] \cap /\mathcal{B}/ \cap \mathcal{D}$. ($\Delta_{ae}$ is the class of all functions derivable a.e.)

f) $ACG_* \oplus H_3$ is a linear subclass of $DB_1 \cap [M_*] \cap /\mathcal{B}/ \cap \mathcal{D}$.

g) $C_n \cap [\mathcal{F}] \cap \mathcal{D}$ and $AC_n \cap [\mathcal{F}] \cap \mathcal{D}$ are linear subclasses of $CM \cap B_1^* \cap [M\ ] \cap T_2 \cap \mathcal{D}$.

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