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SELECTIVE DIFFERENTIATION

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Section 0. Introduction.

The author would like to express gratitude to the various editors of the Real Analysis Exchange who suggested this survey article. He hopes that it will be useful in clarifying some of the underlying principles; in particular, the relationships between selective, biselective and path system derivatives.

A selection can be thought of as either an interval function or a point function. As an interval function, a selection consists of picking one point from the interior of each nondegenerate subinterval $[a,b]$ of R. (Throughout this paper, $[a,b]$ denotes the interval with endpoints a and b even if $a > b$.) However, it is sometimes useful to consider a selection as a point function. Then a selection s is a function whose domain is the upper half plane $U = \{ (x,y): x<y \}$ and which satisfies the relation $x < s(x,y) < y$.

Section 1. History.

Motivation for the basic concepts of selective differentiation came primarily from papers [Gl.,G.-N., N., S.1, S.2, B] due to Gleyzal, Goffman, Neugebauer, Snyder, and Bruckner.

Gleyzal [Gl] said that an interval function ϕ, defined on the collection of all nondegenerate compact subintervals of R, is convergent to a point function f if and only if for each x $\lim_{I \to x} \phi(I) = f(x)$, where $I \to$

x denotes that $x \in I$ and that the measure of I tends to 0. Gleyzal proved that a function $f: R \to R$ is of Baire class 1 if and only if it is the limit of a convergent interval function.

Clearly the comment made above about the two ways of considering a selection applies equally to any interval function. Thus the concept of ϕ being convergent to f is translatable to a statement about the behavior of the point function $\phi: U \to R$ at points close to the boundary of U. That is, for every x we have $f(x) = \lim_{(y,x)\to(x,x)} \phi(y,z)$, where $(y,z) \to (x,x)$, $y \leqq x \leqq z$, $y \neq z$ (the Stolz angle limit). This idea was adopted and researched extensively by Snyder in [S.1, S.2] where he established that various functions such as approximate derivatives are of Baire class 1. (This result was proved earlier by Goffman and Neugebauer in [G.-N.] where they used Gleyzal's idea).

Suppose now that for a given function $f: R \to R$ of Baire class 1 there is an interval function ϕ convergent to f and a selection s such that $f(s[x,y]) = \phi([x,y])$. Then this says even more than that f is of Baire class 1; such an f is also a Darboux function. Indeed this is a characterization of Baire class 1, Darboux functions and was established by Neugebauer in [N.].

There he said that a function f fulfills condition C_1 if and only if there is a selection s such that $\lim_{I \to x} f(s(I)) = f(s)$ for all x. (In [O'M.1] a similar condition was introduced, called C_3, and it was shown that it provides a characterization of M_3 functions according to the classification of Zahorski [Z.].) Neugebauer, in [N.], pushed this idea of "selective continuity" even further and used it to provide a characterization of derivatives. Precisely: a function $f\colon R \to R$ is a derivative if and only if there is a selection s, with respect to which f has the C_1 property, such that $f(s(I)) = |I|$ is an additive interval function; here $|I|$ is the measure of I. It is not hard to imagine how the idea of selective differentiability would arise from this background. Precisely, a function $f\colon R \to R$ is said to differentiable with respect to a selection s if for each x fixed

$$\lim_{h\to 0} \frac{f([x,x+h])-f(x)}{s[x,x+h]-k}$$

exists and is finite. The value of this limit is called the selective derivative of f and denoted as $sf'(x)$.

The final motivation arose out of the work of Bruckner [B] despite the fact that none of his methods involved the notion of selection. Bruckner showed that a question of Zahorski about monotonicity has an affirmative answer. He proved a very interesting theorem which provides a technique for establishing general monotonicity theorems. In 1974, O'Malley (who hates to travel) was ordered by his thesis advisor, Casper Goffman, to visit Bruckner in California. In an effort to obtain an application of Bruckner's technique the selective derivative was introduced and Theorem 17 of [O'M.2] proven. Further thought made it natural to define selective derivates as well. For example, the right upper selective derivate of a function $f\colon R \to R$ at x is

$$\limsup_{k\to 0+} \frac{f(s[x,x+k])-f(x)}{s[x,x+k]-x}\ .$$

Section 2. Examples.

Now we will consider some special selections and we will show what types of differentiation concepts they generate. Our examples are very simple but useful.

Example 1. The lazy selection. For each interval $[a,b]$ set $s[a,b] = \frac{a+b}{2}$. Then the associated derivative is the ordinary derivative and the derivates also correspond to their ordinary counterparts. This simple example brings out an aspect of selective differentiation which will be elaborated on throughout the paper and which is connected with the problem of making a selection so that both endpoints be "satisfied" with the choice. (If a function has a derivative at a point x, then any point selected in the interior of intervals having x as an endpoint will, in a sense, "satisfy" x.)

Example 2. Carrying the idea of "satisfying" one endpoint to its logical extreme, suppose that the function f has at every x at least one finite right derived number. Construct a function g by choosing for each x one of these finite derived numbers as $g(x)$. In addition, pick a sequence $x_1, x_2, \ldots$ such that $x_n \to x$ from above and $\lim\limits_{n\to\infty} \frac{f(x_n)-f(x)}{x_n-x} = g(x)$. Then for each interval $[a,b]$ with $a < b$ select as desired point any term of the sequence associated with a which lies in (a,b). Then f will have g as a right selective derivative. This example shows that one-sided selective differentiation will not give important information either about f or g. It also brings out the idea of path system differentiation and its relation to selective differentiation. Namely, for each x we could consider the path to x to be $\{x_n\}\cup\{x\}$. This will be taken up later in the paper.

Examples 1 and 2 indicate that we need some compatibility of the "desires" of the two endpoints, if the idea of selective differentiation is to be useful. Personifying, we may imagine a meeting attended by all points, where each point has to meet each distinct point and reach an agreement as to the point to be selected. (Obviously, the points at which the function is differentiable need not attend.) The next example will illustrate such a meeting.

Example 3. Suppose that $f: R \to R$ has an approximate derivative g. For each x let E_x be a closed set having density 1 at x such that

$$\lim_{\substack{y \to x \\ y \in E_x}} \frac{f(y)-f(x)}{y-x} = g(x).$$

At a meeting attended by all points clearly each point x wants the selected point to come from E_x, its path. Then the following rule should be adopted by the convention:

Let $a,b \in R$, $a < b$. Let $\alpha = |E_a \cap (a,b)|$, $\beta = |E_b \cap (a,b)|$.

i) If $\min(\alpha,\beta) > \frac{b-a}{2}$, then, obviously, $E_a \cap E_b \cap (a,b) \neq \emptyset$ and the point should be selected from this intersection.

ii) If $\min(\alpha,\beta) \leqq \frac{b-a}{2}$ and $\alpha \neq \beta$, pick the point to "satisfy" the endpoint having the larger of the values α,β.

iii) If $\alpha = \beta \leqq \frac{b-a}{2}$, pick the point from $E_a \cap (a,b)$.

The reason this anticipatory rule should be accepted by all involved is simple. For each value x there is a $\delta > 0$ such that $|E_x \cap (x,x+h)| > |h|/2$, whenever $0 < |h| \leq \delta$. Thus, for any x we will have ultimately either case i) or case ii) so that, ultimately, the selected point will come from E_x. (This also is the essence of the internal intersection property applied to path system differentiation.)

The next example is based on Neugebauer's characterization of derivatives, [N.].

Example 4. Suppose that $F: R \to R$ is differentiable to $f: R \to R$ and that $s[a,b]$ is any point in (a,b) such that $\frac{F(b)-F(a)}{b-a} = f(s[a,b])$. Then f has the C_1 property relative to this selection. Suppose, in addition, that for each x

$$\lim_{h\to 0} \frac{s[x,x+h]-x}{h} = \frac{1}{2}$$

Now, if f has a selective derivative g relative to the selection s, it is easy to verify that g is actually the second Peano derivative of F. However, it is still an open question whether every second Peano derivative is a selective derivative of the first derivative.

Section 3. Bilateral selective derivatives and derivates and biselective derivatives.

One previously undiscussed aspect of selection in bilateral situations is its catalytic nature. This is typified in Lemma 1 of [O'M.2].

Let $f: R \to R$. Let s be a selection, η a positive number and $P = \{x: \frac{f(s[x,x+h]-f(x)}{s[x,x+h]-x} > 0$ for all h with $0 < |h| < \eta\}$. Let, moreover, $x,y \in P$ and $0 < y-x < y$. Then $f(x) < f(y)$.

Then the analysis of f is restricted to P and from that point the selection is not mentioned or needed. This means that the selected points are used only as a bridge, or reagent, between x and y without the behavior of f near or at them being important.

It is this ability of the selection to act as a bridge which yields results such as the following:

i) If $f: R \to R$ has bilateral lower selective derivate positive for all x then f is increasing. (Theorem 1, page 79 [O'M.2])

ii) If $f: R \to R$ has bilateral lower selective derivate not equal to $-\infty$ for any x, then

a) f is measurable and of generalized bounded variation

b) there is a dense open set on which f is differentiable for almost all x. (Theorem 4, page 87 [O'M.2]

iii) If $f: R \to R$ has a selective derivative for all x then

a) there is a sequence of closed sets Q_R, whose union is R, such that f is Lipschitzan relative to Q_k for each k,

b) f is Darboux,

c) the selective derivative is Darboux. (Theorem 11, p. 87 [O'M.2].)

According to iii), every selective derivative is a Darboux function. It is natural to ask whether every selective derivative is of Baire class 1. (Most generalized derivatives are.) However it was shown in [O'M.2] that this is not the case with selective derivatives. Also in [O'M.2] several conditions were provided whereby a selective derivative is of Baire class 2. Such conditions were rendered unnecessary by the work of Miklos Laczkovich in [L.]. He realized an important fact which is reminiscent of

Gleyzal's work. Namely, a selective derivative is not the limit of an interval function but rather the left and right limits of two associated but different interval functions; e.g., $r[a,b] = \frac{f(s[a,b])-f(a)}{s[a,b]-a}$ $(a<b)$. He noted also that the right and left interval functions fulfill the natural relation:

$$(*)\ \min(\ell[a,b],r[a,b]) \leq \frac{f(b)-f(a)}{b-a} \leq \max(\ell[a,b],r[a,b]).$$

Moreover, working in the direction of Gleyzal instead of Neugebauer he realized that it was this right and left convergence and the relation (*) which provided important elements in theorems such as i) - iii) mentioned above. He introduced an ℓ-r derivative using this idea. Among other things, he showed that all ℓ-r derivatives are of Baire class 2 and that every selective derivative is an ℓ-r derivative. He also queried if all ℓ-r derivatives are of honorary Baire class 2. A function h is of honorary Baire class 2, according to Bagemihl and Piranian [B.P.], if there is a function g of Baire class 1 with $\{x: g(x) \neq h(x)\}$ countable. In [O'M.3], an affirmative answer to the query was found. The basis for the solution comes from an investigation of the relation (*). It is deceptively mild and has a simple geometric interpretation. It will be satisfied by a pair of left and right interval functions ℓ,r if and only if the line L through (b,f(b)) with slope $\ell[a,b]$ intersects the line through (a,f(a)) with slope r[a,b] in some point (x,y) with $x \in [a,b]$. This fact led the author [O'M.3] to take a step back to Gleyzal's basic concept combining interval functions and selections.

A biselection is an ordered pair (s,ϕ) where s is a selection and ϕ is any interval function. A function $f: R \to R$ is said to have a biselective derivative bf'(x) relative to the biselection

$b = (s,\phi)$ at x if and only if

$$\lim_{y \to x} \frac{\phi[x,y]-f(x)}{s[x,y]-x} = bf'(x).$$

Clearly what has happened here is that $f(s[x,y])$ has been replaced by $\phi[x,y]$ (the original idea of Gleyzal). It takes little work to establish that there is an equivalence between the concepts of ℓ-r and biselective derivatives. (O'Malley continues to use the latter out of convenience.) As mentioned above, the biselective derivative has the same catalytic nature as the selective in that there is an exact analogue of Lemma 1 of [O'M.2] and iii) a) above also holds. This was established by Laczkovich in [L.]. However neither f nor bf' need be Darboux. This is caused basically by the fact that the biselective derivative need not be a derived number (of f at a given point). Yet the same apparent defect yields that the symmetric derivative of $|x|$ is a biselective derivative. It is an open question precisely which symmetric derivatives are also biselective.

Section 4. Balanced selections and alternate selections.

The fact that a selective derivative is not always of Baire class 1 but is of honorary Baire class 2 suggests three research projects.

The first project.

It is elementary that for a given function $f\colon R \to R$ there can be several different selective derivatives. (This is not the case with such generalized derivatives as the approximate derivative.) Therefore it is possible to conjecture that the fact that a given selective derivatives is strictly of Baire class 2 is simply due so an ill-advised selection and so ask: If a function $f: R \to R$ is selectively differentiable with respect to a selection s and sf' is of Baire class 2, is there an alternate selection t such that f has a selective derivative with respect to t and tf' is of Baire class 1? (A fact that is related but seems neither to help nor to hurt in this investigation is the following Theorem 9, p. 84 [O'M.2]. If $f: R \to R$ is selectively differentiable with respect to both selections s and t, then $\{x: sf'(x) \neq tf'(x)\}$, is countable.) In [O'M.- W.1], the above question was answered. In that paper the relationships between selective differentiation and a strong form of path differentiation called composite differentiation, lead to a counterexample. A function $f: R \to R$ was exhibited which is both selectively differentiable and compositely differentiable and yet every selective derivative of f is strictly of Baire class 2. Thus the first project fails.

The second project

Since a selective derivative may be strictly of Baire class 2, it is natural to seek conditions on a selection s which will automatically force any function selectively differentiable with respect to s to have a selective derivative of Baire class 1. One such condition is based on the idea contained in example 4 of section 2. This condition involves the concept of balance. If, as h lends to 0, the position of $s[x,x+h]$ in $(x,x+h)$ tends to stabilize away from both x and x+h, then any selective derivative with respect to s will be of Baire class 1. To formulate our condition precisely we need the following definitions from [O'M.4]:

Definition. Let I be a compact interval and let $\alpha\in(0,1)$. By the α interval of I we mean the interval of length $\alpha|I|$ with the same center as I.

Definition. A selection s is said to be balanced if there are two positive functions α and δ with the following property: If I is any compact interval having x as one endpoint and if $|I| < \delta(x)$, then s(I) is in the $\alpha(x)$-interval of I.

(In example 4 of section 2 the selection is very heavily balanced in that $\alpha(x)$ can be taken to be as small as predesired.) A perusal of the proof in [O'M.4] reveals that a few small modifications would make the situation apply equally to biselective derivatives. If the first interval function in the ordered pair of a biselection is a balanced selection, then any derivative with respect to this biselection will be of Baire class 1. This also suggests a possibly interesting open question. Suppose that ϕ, the second element of the ordered pair, is, in a weak sense, a selection; more precisely, suppose that for each [a,b] we have

$$\min(f(a),f(b)) \leqq \phi[a,b] \leqq \max(f(a),f(b)).$$

What can be said about biselective derivatives and their primitives with respect to such biselections?

The Third Project

A different attack on the problem goes back to the work of [S.2] and considers a selection as a point function on $U = \{(x,y): x<y\}$. Following his ideas we can define a selective derivative as

$$hv - \lim_{(x,y)\to(x_\emptyset,x_\emptyset)} \frac{f(s(x,y))-f(x_\emptyset)}{s(x,y)-x_\emptyset},$$

where hv - lim indicates that (x,y) approaches $(x_\emptyset,x_\emptyset)$ along the horizontal and vertical line segments in U ending at $(x_\emptyset,x_\emptyset)$.

To develop fully the framework the following definitions are needed:

Definition. For a real number, a, a subset r(a) of U is called a right approach set for a if every (x,y) in r(a) satisfies $a \leq x$ and (a,a) is the only limit point of r(a) in the boundary of U. A left approach set is defined similarly.

Definition. Suppose that for each a in R a right and left approach set, r(a) and $\ell(a)$, have been chosen. The collection C of these sets is said to have the intersection property if for each a there is a $\delta > \emptyset$ such that $r(a_1)\cap\ell(a) \neq \phi$ and $r(a)\cap\ell(a_2) \neq \phi$, whenever $a-\delta < a_1 < a < a_2 < a + \delta$.

Now the idea of selective derivative can be generalized as follows:

Let f: $R \to R$ be fixed and let s be a fixed selection.

Let C be any collection of right and left approach sets having the intersection property. Then f has a selective derivative g(x) relative to s and C if

$$l\pi - \lim_{(x,y)\to(x_\emptyset,x_\emptyset)} \frac{f(s(x,y))-f(x_\emptyset)}{s(x,y)-(x_\emptyset)} = g(x_\emptyset).$$

here lr-lim means (x,y) approaches $(x_\emptyset,x_\emptyset)$ through the right and left approach sets at $x_\emptyset$. In one way this seems a new derivative, however it has been shown in [O'M.5] that:

If f, g, s, and C are as stated above there is a new selection t such that

$$hv - \lim_{(x,y)\to(x_0,x_0)} \frac{f(t(x,y))-f(x_0)}{t(x,y)-x_0} = g(x_0).$$

Still it is possible to use this idea to obtain Baire class 1 selective derivatives. For this another definition is needed.

Definition. Let $0 < \alpha < 1 < \beta$ be two fixed numbers.

Let $C(\alpha,\beta)$ be the collection of right and left approach sets with the property that for each u, $\ell(a)$ is the line segment with slope α ending at (a,a) and r(a) is the line segment with slope β ending at (a,a). Then it follows that, [O'M. 5]:

Let s be a selection and $0 < \alpha < 1 < \beta$ fixed. Suppose $f: R \to R$ and $g: R \to R$ are such that

$$\alpha hv\beta\text{-}\lim_{(x,y)\to(x_0,x_0)} \frac{f(s(x,y))-f(x_0)}{s(x,y)-x_0} = g(x_0)$$

then g is Baire class one. Here $\alpha hv\beta$-lim has obvious meaning of approach to (x_0,y_0) through $C(\alpha,\beta)$ approach sets and the standard hv approach sets.

It is irritating that O'Malley missed an obvious relationship which has bearing on the above theorem. The above theorem requires that f be selectively differentiable with respect to s and also be differentiable relative to these $C(\alpha,\beta)$ approach sets. If only the second of these conditions was satisfied it would appear that the conclusion might fail.

Remember that it is known that there is another selection t such that differentiability with respect to $C(\alpha,\beta)$ becomes selective differentiability with respect to t. On the surface, for the above theorem, that interpretation seems redundant since it is already given that differentiability with respect to s holds. However, a study of how the selection t is determined reveals relatively easily, that t will be balanced. Therefore:

Let s be a selection and let α,β be numbers with $0 < \alpha < 1 < \beta$. Suppose that f and g are functions such that

$$\alpha\beta - \lim_{(x,y)\to(x_0,x_0)} \frac{f(s[x,y])-f(x_0)}{s[x,y]-x} = g(x_0)$$

for each $x_0 \in R$. Then g is of Baire class 1. (This theorem has the same flavor as Corollary 6.3 of [B.O.T.], where differentiation with respect to path systems having the external intersection property is considered.

Next, it seems appropriate to note that it is possible for a function to be selectively differentiable to a function of Baire class 1 where the selection of necessity must be unbalanced. Thus, the following conjecture is false:

A function $f: R \to R$ is selectively differentiable to a function $g: R \to R$ of Baire class 1 if and only if there is a balanced selection t such that $tf' = g$.

Construction of examples exhibiting this behavior can be accomplished using techniques similar to those of [O'M.-W.1], page 35.

Section 5. Path system differentiation and composite differentiation.

We have already encountered several times the idea of path system differentiation (see, e.g., examples 2 and 3 of section 2). For a systematic investigation we need the following definitions [B.O.T.]:

Definition. Let $x \in R$. A path leading to x is a set $E_x \subset R$ such that $x \in E_x$ and x is a point of accumulation of E_x. A system of paths is a collection $\{E_x: x \in R\}$ such that each E_x is a path leading to x. Such a system is bilateral if each x is a bilateral point of accumulation of E_x.

Definition. Let $f: R \to R$ and let $E = \{E_x: x \in R\}$ be a system of paths. If $\lim\limits_{\substack{y \to x \\ y \in E_x}} \frac{f(y)-f(x)}{y-x} = L \in R$, we say that f is E-differentiable at x and we write $f'_E(x) = L$. If f is E-differentiable at every x, then f is simply said to be E-differentiable; f is the E-primitive of f'_E and f'_E is the E-derivative of f.

Definition. Let $E = \{E_x: x \in R\}$ be a system of paths. E will be said to have the various intersection properties described below if there is a positive function δ on R so that, whenever $0 < y-x < \min(\delta(x),\delta(y))$, the sets E_x and E_y intersect in the stated fashion:

Intersection condition: $E_x \cap E_y \cap [x,y] \neq \emptyset$.

Internal intersection condition: $E_x \cap E_y \cap (x,y) \neq \emptyset$

External intersection condition:

$$E_x \cap E_y \cap (y, 2y-x) \neq \emptyset \text{ and } E_x \cap E_y \cap (2x-y, x) \neq \emptyset.$$

There are cases, such as in the study of approximate derivatives, where the path system differentiation is clearly natural. Moreover, the idea of a path system is so broad that it encompasses most other natural forms of differentiation. The only basic requirement is that $f'_E(x)$ is a derived number of f at each point x. Therefore it is not surprising that, as example 2 of section 2 shows, conditions must be placed on the system E to obtain global properties of the E-primitive or E-derivative. The introduction of intersection conditions such as above leads immediately to selective derivatives.

Theorem 3.4 [B.O.T. p.101]

Let E be a system of paths that is bilateral and has the internal intersection property. Then there is a selection s such that every E-differentiable function $f: R \to R$ is selectively differentiable relative to s and $sf' = f'_E$.

The search for a converse of this theorem reveals an interesting unexpected complication in selective differentiation. Suppose that $f: R \to R$ is selectively differentiable with respect to s. The obvious candidate for a path leading to s is the set $E_x = \{y\colon$ there is an $h \neq 0$ with $y = s[x,x+h]\} \cup \{x\}$. It is clear that x is a point of bilateral accumulation of E_x and that $E_{x_1} \cap E_{x_2} \cap (x_1,x_2) \neq \phi$ whenever $x_1 \neq x_2$. Therefore this path system, E is bilateral and fulfills the internal intersection condition. However, f might not be E-differentiable. To see this consider a point x and a sequence $y_1, y_2, \ldots$ converging to x from above such that $y_n \in E_x$ for each n. Let h_n be numbers such that $s[x,x+h_n] = y_n$. We would like to have $\lim_{n\to\infty} \frac{f(y_n)-f(x)}{y_n-x} = sf'(x)$. This will certainly hold if $h_n \to 0$ as $n \to \infty$. But nothing in the definition of the selection s or the path E_x can enforce that condition. If, however, the selection s is balanced and if δ is as in the definition of a balanced selection, we may choose

$E_x = \{y: y=s[x,x+h]$ for some h with $0 < |h| < \delta(x)\} \cup \{x\}$.

Now, if $y_n \in E_x$ and if $y_n \to x$, then the corresponding sequence $\{h_n\}$ converges to 0, so that $f'_E(x) = sf'(x)$.

Another possible choice of paths is to pick a sequence $h_1, h_2, \ldots$ bilaterally converging to 0 and set $E^*_x =$ $\{y: y = s[x,x+h_n]$ for some $n\} \cup \{x\}$. Then the system $E^* =$ $\{E^*_x; x \in R\}$ is bilateral and $f'_{E^*} = sf'$. Yet E^* need not fulfill the internal intersection condition.

It is still an open question whether for every selection s there is a bilateral path system E with the interval intersection property such that $f'_E = sf'$ for each function on f differentiable with respect to s.

Also of interest is the study of path systems fulfilling the intersection condition but not the internal intersection condition. Here biselections come into play. The corresponding theorem is not as general as Theorem 3.4 of [B.O.T.] in that it also depends on the function but it is illuminating. More precisely:

let $f: R \to R$ and let E be a path system with the intersection property. Let f be E-differentiable. Then there is a biselection such that $bf'(x) = f'_E(x)$ for each x.

Proof. Let $E = \{E_x; x \in R\}$ and let δ be a positive function such that for all $x,y \in R$ with $0 < |x-y| < \min(\delta(x),\delta(y))$ we have $E_x \cap E_y \cap [x,y] \neq \emptyset$. We will construct a biselection $b = (s,\phi)$ in a way inspired by example 3 of section 2. Let $x,y \in R$, $x < y$. There are three cases.

a) If $y - x < \min(\delta(x),\delta(y))$ and $E_x \cap E_y \cap (x,y) \neq \emptyset$ pick any point z from this intersection and set $s[x,y] = z$, $\phi[x,y] = f(z)$.

b) If $y - x$ $\min(\delta(x),\delta(y))$ and $E_x \cap E_y \cap [x,y] \subset \{x,y\}$, we proceed as follows:

Suppose first that $x \in E_y$. Let $\frac{f(y)-f(x)}{y-x} = \alpha$ and consider over the interval (x,y) the region between the lines through $(y,f(y))$ of slopes $\alpha-(y-x)$, $\alpha + (y-x)$, respectively. This region has nonempty intersection with the line through $(x,f(x))$ of slope $f_E'(x)$. Now pick any point (u,v) in this intersection and set $s[x,y] = u, \phi[x,y] = v$.

If $x \notin E_y$, then $y \in E_x$ and we proceed similarly.

c) If $y - x \geqq \min(\delta(x),\delta(y))$ and $\delta(x) \geqq \delta(y)$, set $x = e$ and $s[x,y] = u$, $\phi[x,y] = v$, where (u,v) is the midpoint of the segment over the interval (x,y) of the line through $(e,f(e))$ with slope $f_E'(e)$. If $y - x \geqq \min(\delta(x), \delta(y))$ and $\delta(x) < \delta(y)$, we proceed in the same way taking $e = y$.

It is easy to prove that (with $b = (s,\phi)$) we have $bf' = f_E'$.

It should be noticed that if, at every x, $f_E'(x)$ is a bilateral derived number of f, then a little case in the preceding proof would yield that a f_E' will become a selective derivative. In fact, f_E' will be a path derivative as the following theorem indicated.

Theorem. Let $f: R \to R$ and let E be a path system with intersection property. Suppose that f is E-differentiable and that, for each x, $f_E'(x)$ is a bilateral derived number of f. Then there is a bilateral path system E^* with the internal intersection property such that $f_E' = f_{E^*}'$. (Note that f_E' is also a selective derivative of f.)

As corollaries we obtain improvements of several theorems in [B.O.T]:

New 6.4 Theorem. If a bilateral path system E satisfies the intersection condition, then every E-derivative has the Darboux property.

New 6.8 Corollary: If a bilateral path system E satisfies the intersection condition, then every E-derivative has the Denjoy property.

(It was shown in [0.M.6] that every selective derivative has the Denjoy property.)

New 8.1 Theorem. Let E be a nonporous path system satisfying the intersection condition. Let f be an E-differentiable function and let f'_E attain the values M and -M on an interval $I_\emptyset$. Then there is a subinterval I of $I_\emptyset$ on which f is differentiable and f' attains both values M and -M. (This new theorem uses the fact that every selective derivative is honorary Baire class 2.) See also [O'M.-W.2]

Further results such as 4.2 Theorem can be deduced from the corresponding results for biselective differentiation.

The point to be made here is that the intersection condition enables us to exploit many ideas of selective or biselective differentiation for the investigation of path derivatives. However, there are several ideas which are more natural to path differentiation. Perhaps the most important of these is the concept of nonporous path system. It yields the M_3 property of E-derivatives (according to classification of Zahorski [Z.]). In addition we get the theorem that if E is a nonporous path system, then every E-differentiable monotone function is differentiable. It is not easy to obtain similar results for biselective derivatives and primitives.

A further topic to be discussed along these lines is the composite derivative mentioned above. We need the following definitions (see [O'M.-W.1]):

Definition. A decomposition (of R) is any sequence of closed sets, E_n, with union R.

Definition. A function f is said to have the function g as composite derivative relative to the decomposition E_1, E_2, ... if for each n and each $x \in E_n$ we have

$$\lim_{\substack{y \to x \\ y \in E_n}} \frac{f(y)-f(x)}{y-x} = g(x).$$

If x is an isolated point of E_n, this relation is considered to hold vacuously.

Composite differentiability can be thought of as a type of "uniform path system differentiability" if, for each x, there is an n(x) such that x is a limit point of $E_{n(x)}$. Then we can simply set $E_x = E_{n(x)}$ for each $x \in R$. Even without this condition it is possible to prove:

3.1 Theorem [O'M-W.1]. Let the sets E_1, E_2, ... form a decomposition of R, and let f and g be functions. Suppose that g is a composite derivative of f relative to E_n. Then there is a biselection b such that $bf' = g$.

Further: [O'M.-W.1]

3.2 Theorem. Let E_n, f, and g be as in 3.1. Suppose, in addition, that for each x, g(x) is a bilateral derived number of f at x. Then there is a selection s such that $sf' = g$.

It is possible to find conditions on either the biselection or the path system under which the corresponding derivative becomes a composite derivative. For biselective derivative the reader is referred to [O'M.-W.1] for the details. For the path system derivative the key condition takes the form of a strong intersection condition. This is described in [O'M.7].

Section 6. Related ideas.

Recently U. Laczkovich and Y. Pokorný have investigated several problems related to selective and biselective derivatives. In the interesting paper [L.P.] they show:

1) There is a continuous function f and a selection s such that the finite selective derivative sf' exists everywhere and sf' is not of Baire class 1 on any interval. (Such a selection must be non-balanced on every interval.)

This theorem shows that no significant improvement of the result about the honorary Baire class 2 is possible.

2) For every $f: R \to R$ and every biselection b the measure of the set $\{x: bf'(x) = \infty\}$ is zero. (Here, of course, the biselection is allowed to generate an infinite derivative.)

3) If the definition of selective derivative is generalized in the obvious way to introduce the concept of an approximate selective derivative, we get a situation similar to Example 2 of section 2. More precisely:

Under the continuum hypothesis there is a function f such that for every $g: R \to R$ there is a selection s fulfilling the relation

$$\operatorname*{ap-lim}_{y \to x} \frac{f(s[x,y])-f(x)}{s[x,y]-x} = g(x) \text{ for each } x \in R.$$

E. Lazarow, [La] in a similar vein, proved the following theorem.

Let $f: R \to R$ be Lebesgue measurable. Then there is a selection s and a set P of cardinality c such that f has a selective derivative (possibly infinite) with respect to s at each point of P. Also:

There is a continuous function $f: [0,1] \to R$ such that for every selection s the set of points at which the selective derivative (possibly infinite, of f with respect to s exists is of measure zero and of first category. In fact the set of such functions is a residual subset of $C[0,1]$.

In another direction the interested reader is referred to a recent paper of Bruckner, Laczkovich, Petruska and Thomson [B.L.P.T.]. In that paper the sequential derivative, a certain type of path derivative, is considered. The authors investigate conditions under which the existence of a sequential derivative a.e. in a set A implies the existence of the approximate derivative a.e. in A. They also investigate the additional question when the sequential derivative equals the approximate derivative. Their work, however, is very technical and too complex to be reproduced here.

References

[B.P.] F. Bagemihl and G. Piranian, Boundary functions for functions defined in a disk, Mich. Math. J., 8 (1961), 201-207.

[B.] A. M. Bruckner, An affirmative answer to a problem of and some consequences, Mich. Math. J., 13 (1966), 15-26.

[B.L.P.T] A. M. Bruckner, M. Laczkovich, G. Petruska, B. S. Thomson, Porosity and Approximate Derivatives, to appear.

[B.O.T.] A. M. Bruckner, R. J. O'Malley, B. S. Thomson, Path Derivatives: A Unified View of Certain Generalized Derivatives, T.A.M.S., 283 (1984), 97-125.

[Gl.] A. Gleyzal, Interval functions, Duke Math. J. 8 (1941), 223-230.

[G.-N.] C. Goffman and C. Neugebauer, On approximate derivatives, P.A.M.S. 11 (1960), 962-966.

[L.] M. Laczkovich, On the Baire Class of selective derivatives, Acta. Math. Acad. Sci. Hungar., 29 (1977), 99-105.

[L.P.] M. Laczkovich, I. Pokorný, Selective and bi-selective derivatives, to appear.

[La.] E. Lazarow, Selective differentiation of typical continuous functions, Real Anal. Ex., 9 (1983-84), 463-472.

[N.] C. Neugebauer, Darboux functions of Baire Class 1 and derivatives, P.A.M.S., 13 (1962), 838-843.

[O'M.1] R. J. O'Malley, M_3 function, Indiana U. Math. J., 24 (1974), 585-591.

[O'M.2] Ibid., Selective Derivatives, Acta. Math. Acad. Sci. Hungar., 29 (1977), 77-97.

[O'M.3] Ibid., Bi-selective derivatives are of honorary Baire class Acta Math. Acad. Sci. Hungar., 41 (1983), 111-117.

[O'M.4] Ibid., Balanced Selections, Real Anal. Ex., 8 (1982-83), 504-508.

[O'M.5] Ibid., Selective Differentiation: Redefining Selections, Acta. Math. Sci. Hungar., 40 (1982), 39-44.

[O'M.6] Ibid., Selective Derivatives, M_2 and the Denjoy-Clarkson property, Acta. Math. Acad. Sci. Hungar. 36 (1980), 195-199.

[O'M.7] Ibid., The Multiple Intersection property for path derivatives, to appear in Funda. Math. 1986.

[O'M.-W.1] R.J. O'Malley, C. E. Weil, Selective, Bi-selective, and Composite Differentiation, Acta. Math. Acad. Sci. Hungar., 43 (1984), 31-36.

[O'M.-W.2] R. J. O'Malley, C. E. Weil, The oscillatory behavior of certain derivatives, T.A.M.S., 234 (1977), 467-481.

[S.1] L. Snyder, Approximate Stolz Angles, P.A.M.S., 17 (1966), 416-422.

[S.2] Ibid., Continuous Stolz extensions and boundary functions, T.A.M.S., 119 (1965), 417-427.

[Z.] Z. Zahorski, Sur la première dérivée, T.A.M.S., 69 (1950), 1-54.