A New Approach to Integration

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1. Introduction. Recently ([7], [8], [10], [13]) a new theory of integration has been developed. It is more general than the Lebesgue integral [3] but it retains the monotone and dominated convergence properties of that integral. Unlike the Riemann and Lebesgue integrals, it always retrieves a function from its everywhere finite derivative:

$$\int_a^b F'(x)dx = F(b) - F(a).$$

In fact, slightly more is true (see Theorem 2.1 below).

This "new" integral is really not new - it is equivalent to the Perron and the restricted Denjoy integrals introduced early this century ([18] and [7]), i.e., a function is integrable in the new sense iff it is Perron and restricted Denjoy integrable, in which case the values of the integrals are all the same. What is new and remarkable is the definition of this integral which is very elementary, namely, a slight modification of Riemann's. In accordance with present convention we shall refer to this integral as "generalized Riemann integral". The status of Lebesgue integrability is easily described: f is Lebesgue integrable iff both f and $|f|$ are generalized Riemann integrable, in which case the definite integral of f in its two meanings yields the same number. It follows that a non-negative real generalized Riemann integrable function is Lebesgue integrable.

Since the new integral integrates every derivative while the Lebesgue integral does not, the class of Lebesgue integrable functions is a proper subset of the class of generalized Riemann integrable functions.

Although this new approach was initiated by J. Kurzweil [10], full credit for an independent discovery and extensive development of the theory must go to R. Henstock ([7], [8]).

2. Definition and an important property of the integral.
We enunciate the theory for a real valued function of a real variable, although a corresponding theory holds in more general settings (e.g., for mappings of a closed interval into real Euclidean n-spaces).

Let $f(x)$ be defined on $[a,b]$, $-\infty < a < b < \infty$.
The Riemann integral of f may be introduced as follows:

If there exists a real number I so that, for each $\epsilon > 0$, there is a constant $\delta > 0$ such that

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \epsilon$$

whenever $a = x_0 < x_1 < \ldots < x_n = b$ _and_ $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta$ _for_ $k = 1, 2, \ldots, n$, _then such an_ I _is unique and is the Riemann integral of_ f _on_ $[a,b]$.

If instead of demanding that the partition be of constant 'fineness' as above, we allow the fineness to vary depending on the behaviour of the function, i.e., from point to point in $[a,b]$, we obtain the definition of the generalized Riemann integral: _If there exists a real number_ I _so that, for each_ $\epsilon > 0$, _there is a positive function_ $\delta_\epsilon(t)$ _on_ $[a,b]$ _such that_

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \epsilon$$

whenever $a = x_0 < x_1 < \ldots < x_n = b$ _and_ $x_{k-1} \leqslant t_k \leqslant x_k$, $x_k - x_{k-1} < \delta_\epsilon(t_k)$ _for_ $k = 1, 2, \ldots, n$, _then such an_ I _is unique and is called the generalized Riemann integral of_ f _on_ $[a,b]$, _denoted by_ $\int_a^b f(x)dx$ _or_ $\int_a^b f$.

Here is a (known) sample of the theory of that integral.

Theorem 2.1. _Let_ $-\infty < a < b < \infty$ _and let_ F _be a real function, continuous on_ $[a,b]$ _and (finitely) differentiable on_ (a,b).

Then

$$(2.1)\qquad \int_a^b F'(x)dx = F(b)-F(a),$$

the integral being generalized Riemann. (F' is extended to $[a,b]$ arbitrarily, e.g., by setting $F'(a) = F'(b) = 0$).

Proof: Let $\epsilon > 0$ be given. Choose, for every $t \in (a,b)$, a number $\delta_\epsilon(t)$ such that

$$(2.2)\qquad |F(x) - F(t) - F'(t)(x-t)| < \eta|x-t|,\ \eta = \epsilon/(4+b-a),$$

whenever $0 < |x-t| < \delta_\epsilon(t)$. Also choose for $t = a$ and $t = b$, a number $\delta_\epsilon(t)$ such that

$$(2.3)\qquad |F(x) - F(t)| < \eta,\ |F'(t)(x-t)| < \eta$$

whenever $|x-t| < \delta_\epsilon(t)$, $x \in [a,b]$.

Let $a = x_0 < x_1 < \dots < x_n = b$ and $x_{k-1} \leq t_k \leq x_k$, $x_k - x_{k-1} < \delta_\epsilon(t_k)$ for $k = 1, 2, \dots, n$. Let $1 \leq k \leq n$. If $a < t_k < b$, then, by (2.2),

$$(2.4)\qquad |F(x_k) - F(x_{k-1}) - F'(t_k)(x_k-x_{k-1})| \leq$$

$$|F(x_k) - F(t_k) - F'(t_k)(x_k-t_k)| + |F(x_{k-1}) - F(t_k) - F'(t_k)(x_{k-1}-t_k$$

$$< \eta(x_k-t_k) + \eta(t_k-x_{k-1}) = \eta(x_k-x_{k-1}).$$

If t_k is a or b, then the left-most member of (2.4) is less than 2η by (2.3). Hence

$$\left|F(b) - F(a) - \sum_{k=1}^{n} F'(t_k)(x_k-x_{k-1})\right| \leq \sum_{k=1}^{n}\left|F(x_k)-F(x_{k-1})-F'(t_k)(x_k-x_{k-1})\right|$$

$$< 4\eta + \sum_{k=1}^{n}\eta(x_k-x_{k-1}) = \epsilon,$$

and (2.1) is proved.

3. The integral in action. The usefulness of the property (2.1) can be seen from the following proof of a strong form of Taylor's theorem; (cf. [9], §143).

Theorem 3.1 Let $-\infty < a < b < \infty$ and let F be a real function, continuous at b. Let n be a positive integer and let F have a (finite) nth derivative throughout (a,b) with $F^{(n-1)}$ Then, for some ξ, $a < \xi < b$, we have

$$(3.1) \qquad F(b) = \sum_{k=0}^{n-1} \frac{F^{(k)}(a)}{k!}(b-a)^k + \frac{F^{(n)}(\xi)}{n!}(b-a)^n.$$

Note: Unlike the usual forms of Taylor's theorem, no differentiability condition is imposed on F at b, which is natural, as no derivative of F at b of order greater than zero appears in (3.1).

Proof: We shall use some of the simplest facts about integrals. Consider the number

$$(3.2) \qquad I = \int_a^b \int_a^{t_n} \cdots \int_a^{t_3} \int_a^{t_2} F^{(n)}(t_1)\,dt_1 dt_2 \ldots dt_n$$

which, of course, means $\int_a^b F'(t_1)dt_1$ if $n=1$.

By repeated integrations and Theorem 2.1,

$$(3.3) \qquad I = F(b) - \sum_{k=o}^{n-1} \frac{F^{(k)}(a)}{k!}(b-a)^k .$$

Taking $F(x) = (x-a)^n/n!$, we have by (3.2) and (3.3)

$$(3.4) \qquad I_o = \int_a^b \cdots \int_a^{t_2} dt_1 \ldots dt_n = \frac{(b-a)^n}{n!}.$$

Suppose I/I_o were smaller than every $F^{(n)}(x)$, $a < x < b$.

Then $F^{(n)}(x) - II_o^{-1}$ would be positive on (a,b) and we would get

$$0 = \int_a^b \int_a^{t_n} \cdots \int_a^{t_2} \left[F^{(n)}(t_1) - II_0^{-1}\right] dt_1 \ldots dt_n > 0.$$

Thus I/I_0 is $\geqslant$ some $F^{(n)}(\alpha)$, $a < \alpha < b$ and similarly I/I_0 is $\leqslant$ some $F^{(n)}(\beta)$, $a < \beta < b$. By the well known Darboux property of derivatives, there is ξ, $a < \xi < b$, with $I/I_0 = F^{(n)}(\xi)$. By (3.3) and (3.4) we have (3.1).

4. Improper integrals. The new approach to integration may be used to characterize improperly Riemann integrable functions and a certain class of such functions for which very general quadrature formulas converge to the integral.

By the usual convention, a real function f defined on $(a,b]$, with $-\infty < a < b < \infty$, is improperly Riemann integrable on $(a,b]$ iff f is Riemann integrable on $[s,b]$ for each s with $a < s < b$, and $\lim_{s \to a+} \int_s^b f$ exists (finite).

Improperly Riemann integrable functions, while not necessarily Lebesgue integrable, are generalized Riemann integrable. In fact, the improper Riemann integral is exactly the generalized Riemann integral with $\delta_\epsilon(t)$ non-decreasing on $(a,b]$ for each $\epsilon > 0$ ([12], Theorem 2).

If in the definition of the generalized Riemann integral on $[0,1]$, $\delta_\epsilon(t)$ has to be linear on $(0,1]$ for each $\epsilon>0$, the integral so obtained is the <u>dominated integral</u> of Osgood and Shisha ([16] and [17]), an improper integral for which every sequence of 'reasonable' quadrature formulas converges to it ([12], Theorem 1).

A slight modification of the finite interval definition yields the generalized Riemann integral on $[0,\infty)$:

If there exists an I *so that corresponding to every* $\epsilon > 0$ *there is a positive function* $\delta_\epsilon(t)$ *on* $[0,\infty)$ *and a positive number* $B(\epsilon)$ *such that*

$$\left| I - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \right| < \epsilon$$

whenever $\quad 0 = x_0 < \dots < x_n, x_n > B(\epsilon),$

and $\quad x_{k-1} \leq t_k \leq x_k,\ x_k - x_{k-1} < \delta_\epsilon(t_k)$

for $k = 1, 2, \dots, n,$ *then* I *(necessarily unique) is called the generalized Riemann integral of* f *on* $[0,\infty)$.

If in the last definition, for every $\epsilon > 0$, $\delta_\epsilon(t)$ is a constant, the resulting integral on $[0,\infty)$ is the <u>simple integral</u> of Haber and Shisha ([4] and [5]). Hence the relationship $\longrightarrow$

of simple integrability to generalized Riemann integrability on $[0, \infty)$ is the same as the relationship of proper Riemann integrability to generalized Riemann integrability on intervals $[a,b]$, $-\infty < a < b < \infty$. Thus from the perspective of the generalized Riemann integral, the simple integral rather than the improper Riemann integral $\int_0^\infty$ seems to be the natural extension of the Riemann integral to $[0,\infty)$.

5. <u>Concluding remarks</u>. It is remarkable that the generalized Riemann integral was arrived at so late, particularly when one notes the great effort which was expended in the early years of the century to describe the Lebesgue integral as a limit of Riemann-type sums. A result in this direction was proved by Lebesgue in 1909 [11] when he showed that his integral could be approximated by such sums. Moreover, Borel in 1910 ([1] and [2]), Hahn in 1914 [6], and others, defined integrals more general than Riemann's as limits of Riemann-like sums.

For completeness it should be noted that a slight modification of the definition of the generalized Riemann integral yields the Lebesgue integral itself ([14] and [15]).

Due to both its power and simplicity, the generalized

Riemann integral should become thoroughly familiar to all real analysts. It can also be used as an easy method for initiating students to a serious study of integration.

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