Michael J. Evans, Department of Mathematics, North Carolina State University, Raleigh, NC 27695-8205

HIGH ORDER SMOOTHNESS

In the talk given at the Louisville Symposium and in this summary of that presentation, all functions are assumed to be Lebesgue measurable real valued functions defined on the real line $\mathbb{R}$. The classical notion of smoothness is that f is smooth at x provided

$$f(x+t)+f(x-t)-2f(x) = o(t) \text{ as } t \to 0,$$

and f is called a smooth function if it is smooth at each point $x \in \mathbb{R}$.

The following theorem is a summary of several known properties of smooth functions. It is essentially due to C. J. Neugebauer [4], although several of its parts have been proved by earlier authors using more restrictive hypotheses.

Theorem A. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be smooth. Then*

a) f *is in class Baire** *one (every perfect set* P *contains a portion* Q *such that the restriction of* f *to* Q *is continuous.)*

b) *if* $E = \{x: f'(x)$ *exists and is finite*$\}$, *then* E *is c-dense.*

c) *if* f *has the Darboux property on* $\mathbb{R}$, *then*

i) f' *has the Darboux property on* E

ii) *if* $f'(x) \geq 0$ *for each* x *in* E, *then* f *is increasing and continuous on* $\mathbb{R}$.

The notion of smoothness naturally extends to the $L_p (1 \leq p < \infty)$ setting by saying that f is L_p smooth at x provided

$$\left\{\frac{1}{h}\int_0^h |f(x+t)+f(x-t)-2f(x)|^p dt\right\}^{\frac{1}{p}} = o(h) \text{ as } h \to 0$$

and f is called an L_p smooth function provided that f is L_p smooth at each real number x.

Neugebauer [4] and O'Malley [6] have established the following analog to Theorem A.

Theorem B. Let $f: \mathbb{R} \to \mathbb{R}$ be L_p smooth. Then

a) f is in class Baire* one

b) if $E_p = \{x: f'_{L_p}(x)$ exists and is finite$\}$, then E_p is c-dense

c) if f has the Darboux property on $\mathbb{R}$, then

i) f'_{L_p} has the Darboux property on E_p

ii) if $f'_{L_p}(x) \geq 0$ for each x in E_p, then f is nondecreasing and continuous on $\mathbb{R}$.

Turning now to the high order setting, we say that f has a k^{th} Peano derivative at x if there is a polynomial $Q_{x,k}(t)$ of degree at most k such that $Q_{x,k}(0) = f(x)$ and

$$f(x+t) - Q_{x,k}(t) = o(t^k) \text{ as } t \to 0,$$

and the value of this k^{th} Peano derivative at x is denoted by $f_k(x)$, where $f_k(x)/k!$ is the coefficient of t^k in $Q_{x,k}(t)$.

If there is a polynomial $P_{x,k}(t)$ of degree at most k for which

$$\frac{1}{2}[f(x+t)+(-1)^k f(x-t)] - P_{x,k}(t) = o(t^k) \text{ as } t \to 0,$$

then f is said to have a k^{th} symmetric derivative (frequently called a k^{th} derivative in the sense of de la Vallee Poussin) at x and the value of this derivative at x is denoted by $D^k f(x)$, where $D^k(f)x/k!$ is the coefficient of t^k in $P_{x,k}(t)$. (If k is even, we shall further require that $P_{x,k}(0) = f(x)$.)

Following T.K. Dutta [1], we now define high order smoothness in the following manner. Suppose that m is a natural number greater than or equal to 2 and that f has an m-2 symmetric derivative at x. We say that f is m-smooth at x provided,

$$\frac{1}{2}[f(x+t) + (-1)^m f(x-t)] - P_{x,m-2}(t) = o(t^{m-1}) \text{ as } t \to 0,$$

and f is said to be m-smooth if it is m-smooth at each x in $\mathbb{R}$. The following result, a high order analog of Theorem A, is primarily due to Dutta [1], with the hypotheses and conclusions sharpened to their current state by the present author in [2] and [3].

<u>Theorem A^1.</u> <u>Let</u> $f: \mathbb{R} \to \mathbb{R}$ <u>be m-smooth.</u> <u>Then</u>

a) $D^{m-2}f$ <u>is in class Baire* one</u>

b) $f^{(m-2)}$ <u>exists finitely on an open dense set and is continuous there</u>

c) <u>if</u> $E = \{x: f_{m-1}(x)$ <u>exists and is finite</u>$\}$, <u>then</u> E <u>is c-dense</u>

d) <u>if</u> f_{m-2} <u>exists on</u> $\mathbb{R}$ <u>and</u> f <u>has the Darboux property on</u> $\mathbb{R}$, <u>then</u>

   i) f_{m-1} <u>has the Darboux property on</u> E

   ii) <u>if</u> $f_{m-1}(x) \geq 0$ <u>for each</u> x <u>in</u> E, <u>then</u> f_{m-2} <u>is non-decreasing and continuous on</u> $\mathbb{R}$.

The notion of m-smoothness can, of course, be carried over to the L_p setting in the obvious manner and the result analogous to Theorem B is considered in [3].

References

1. T. K. Dutta, Generalised smooth functions, Acta Math. Acad. Sci. Hungar, 40(1982), 29-37.

2. M. J. Evans, High order smoothness, (submitted).

3. ________, Peano differentiation and high order smoothness in L_p, Bull. Inst. Math. Acad. Sinica (to appear).

4. C. J. Neugebauer, Symmetric, continuous, and smooth functions, Duke Math. J. 31(1964), 23-32.

5. ____________, Smoothness and differentiability in L_p, Studia Math. 25(1964), 81-91.

6. R. J. O'Malley, Baire* 1 Darboux functions, Proc. Amer. Math. Soc. 60(1976), 187-192.