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TWO MORE CHARACTERIZATIONS OF BESOV-BERGMAN-LIPSCHITZ SPACES

Dedicated to Francisco Vieira de Sales on his 100th Birthday

In the early 1960's the following spaces were introduced, now known as Besov spaces. For $0 < \alpha < 1$, $1 < r, s < \infty$, let

$$\Lambda(\alpha,r,s) = \{f:[-\pi,\pi] \to R \; ; \; \|f\|_{\Lambda(\alpha,r,s)} = \|f\|_r + \left(\int_{-\pi}^{\pi} \frac{(\|f(x+t)-f(x)\|_r)^s}{|t|^{1+\alpha s}} dt\right)^{1/s} < \infty\}.$$

where $\| \; \|_r$ is the Lebesgue space L^r-norm. For these spaces the reader is referred to [1], [7], [8] and [9].

Notice that $\Lambda(\alpha,\infty,\infty)$ is the usual Lipschitz spaces.

The following spaces of analytic functions on the disk have been studied in depth by several people, for example by E. Stein, M. Taibleson, A. L. Shieds, and others.

$$J^p = \{g: D \to C \text{ , Analytic, } \|g\|_{J^p} = |g(0)| + \frac{1}{\pi} \int_0^1 \int_{-\pi}^{\pi} |g'(re^{i\theta})| (1-r)^{\frac{1}{p} - 1} d\theta dr < \infty\}$$

for $1 \leq p < \infty$. The dash means derivative.

In [9], M. Taibelson has shown that $\Lambda(1 - \frac{1}{p}, 1, 1)$ is equivalent as Banach spaces to J^p for $1 < p < \infty$.

In these notes we propose to give two very simple characterizations of the spaces $\Lambda(1 - \frac{1}{p}, 1, 1)$ and J^p for $1 < p < \infty$, in terms of non-increasing functions. For $p = 1$ we also get a result.

Consider the interval $[a-h,a+h] \subset [-\pi,\pi]$ where $h > 0$. Let $g:(a,a+h] \to R$ be a positive, non-increasing function, let $\overline{f}$ be the "even" extension of g on the interval $[a-h,a+h]$, and 0 elsewhere. And let $\overline{h}$ be the "odd" extension of g on the interval $[a-h,a+h]$ and 0 elsewhere.

Define the spaces I^p for $1 < p < \infty$ by $I^p = \{f:[-\pi,\pi]\to R\ ;$ $f(t) = \sum_{i=1}^{\infty} f_i(t)$ such that $\sum_{i=1}^{\infty} \|f_i\|_{L(p,1)} < \infty\}$ where f_i's are of type $\overline{f}$ above, restricted to the interval $I_i = [x_i - h_i, x_i + h_i]$. (See figure 1).

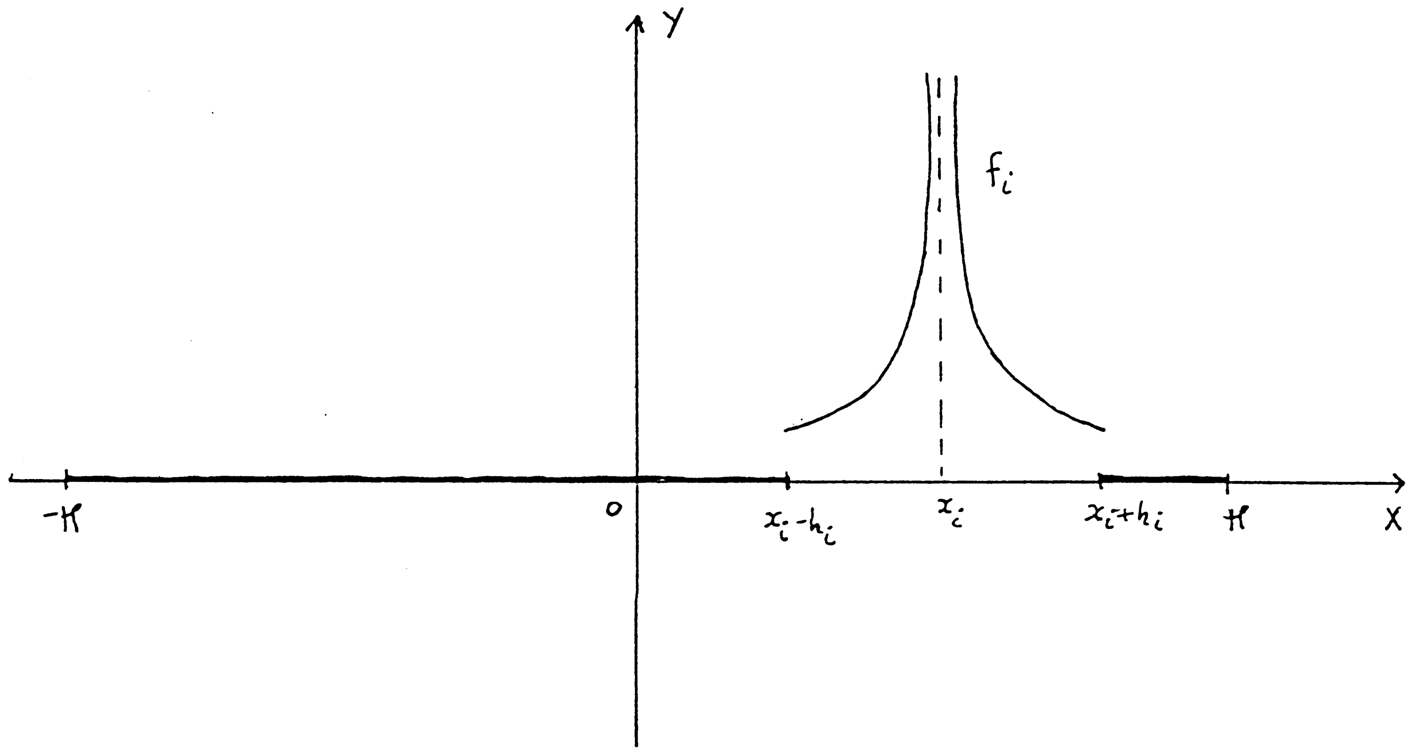

figure 1

Endow I^p with the norm $\|f\|_{I^p} = \operatorname{Inf} \sum_{i=1}^{\infty} \|f_i\|_{L(p,1)}$ where the infimum is taken over all possible representations of f.

Also we define the space G^p as I^p when the f_i's are replaced by h_i's and the h_i's are all of type $\overline{h}$ above on $I_i = [x_i - h_i, x_i + h_i]$. (see figure 2).

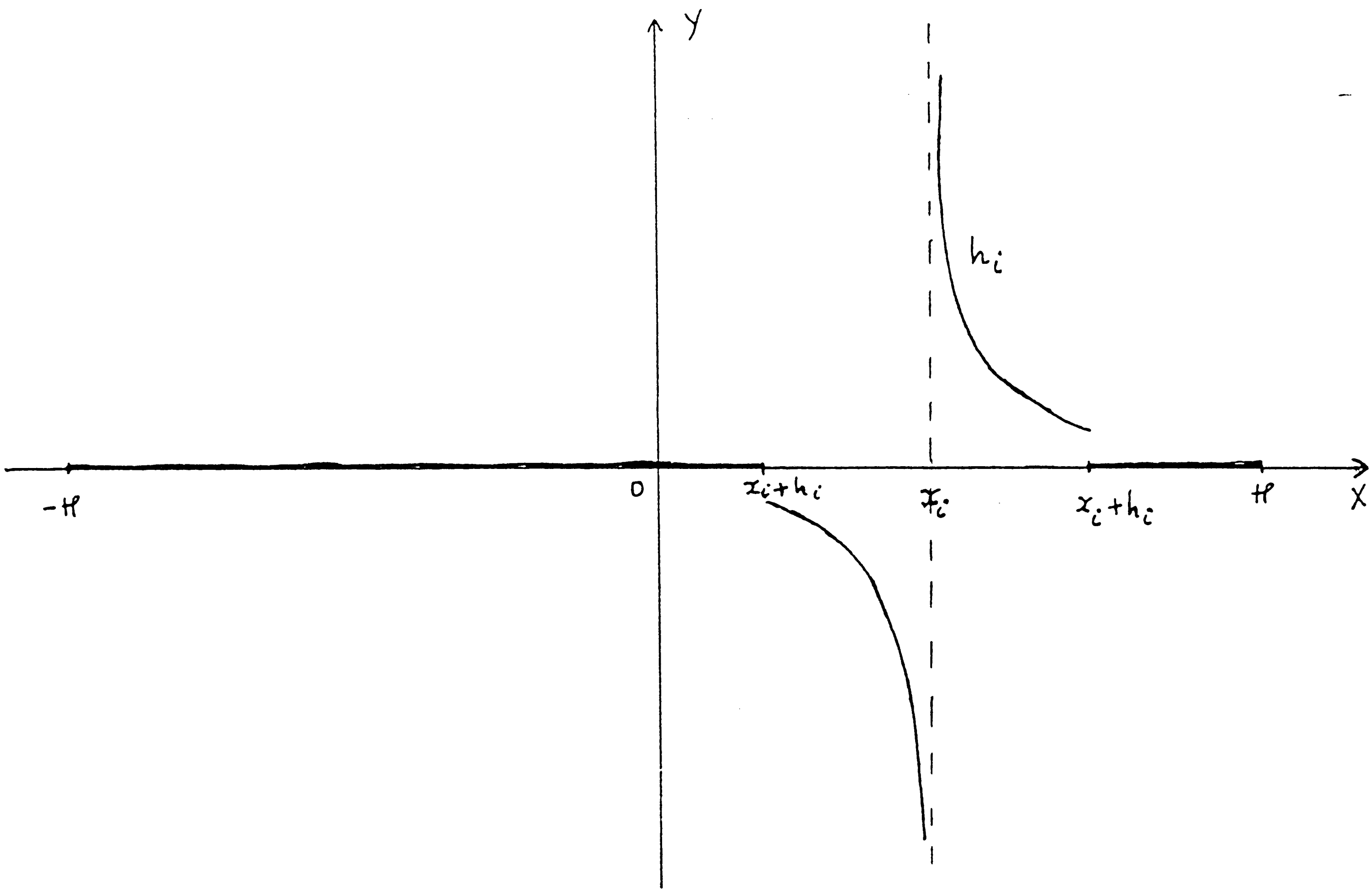

figure 2.

Note $f \in L(p,1)$ if and only if $\|f\|_{L(p,1)} = \frac{1}{p}\int_0^\infty f^*(t)t^{1/p}\frac{dt}{t} < \infty$,

where f^* is the decreasing rearrangement of f, defined by

$f^*(t) = \operatorname{Inf}\{y>0,\ m(f,y) \le t\}$, $t > 0$, $m(f,y) = \left|\{x: |f(x)| > y\}\right|$, the outside bars mean the Lebesgue measure of the indicated set. Recall $L(p,1)$ is called Lorentz spaces.

We have the following result.

THEOREM A. The spaces I^p, G^p, J^p and $\Lambda(1 - \frac{1}{p}, 1, 1)$ are the same in the sense of Banach spaces, with equivalent norms, for $1 < p < \infty$.

Notice the sense meant in this theorem is that $f \in G^p$ if and only if $f \in \Lambda(1 - \frac{1}{p}, 1, 1)$; moreover, $M\|f\|_{I^p} \le \|f\|_{\Lambda(1-\frac{1}{p},1,1)} \le N\|f\|_{I^p}$ for some absolute constants M and N.

In case of J^p we consider the non-tangencial limit that is $f \in I^p$ if and only if $F \in J^p$, moreover the I^p and J^p are equivalent, where

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} f(t)dt.$$

In case of $p = 1$, we have to redefine the space I^1, in order to get the equivalence with J^1. In fact $I^1 = \{f:[-\pi,\pi] \to \mathbb{R} \; ; \; f(t) = \sum_{i=1}^{\infty} f_i(t)$ such that $\sum_{i=1}^{\infty} \|f_i\|_{L\log^+L} < \infty\}$, where $f \in L \log^+ L$ if and only if $\int_{-\pi}^{\pi} |f| \log^+ |f| < \infty$.

$$\log^+ x = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ \log x & \text{if } x > 1 \end{cases}$$

We have the following.

THEOREM B. $f \in I^1$ if and only if $F \in J^1$, moreover the norms are equivalent.

In case of G^1 we could exchange the definition of $\|f\|_{G^1}$ above by $\|f\|^*_{G^1} = \text{Inf} \sum_{i=1}^{\infty} \|h_i\|_{ReH^1}$, where the infimum is taken over all possible representations of f and $f \in ReH^1$ if f is the boundary value of an analytic function F in H^1, that $F \in H^1$ if and only if

$$\|F\|_{H^1} = \lim_{r \to 1} \int_{-\pi}^{\pi} \left|F(re^{i\theta})\right| dr < \infty.$$

Then for G^1 with this description we have,

THEOREM C. $f \in G^1$ if and only if $F \in J^1$ with equivalent norms.

We would like to point out that in Theorem A, if we start off with a function g defined by (C a constant),

$$g(x) = \frac{C}{(2h)^{1/p}} \chi_{(0,h]}(t) \quad \text{then} \quad \bar{h}(x) = \frac{C}{(2h)^{1/p}} \left[-\chi_{[-h,0)}(x) + \chi_{(0,h]}(x) \right]$$

is a special atom. They were introduced by the author in [2], [3], [4], [5] and [6]. Thus Theorem A generalizes the main theorem in [6].

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Professor Piotrowski's report will appear as a Topical Survey in a later issue of the *Real Analysis Exchange*.