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Infinite Peano derivatives

Recall that a function $f:\mathbb{R}\quad\mathbb{R}$ has a (finite) n th Peano derivative at x means that there are numbers $f\ (x), f\ (x), \cdots {}_n f\ (x)$ such that

(1) $f(x + h) = f(x) + hf\ (x) + \cdots + h^n f(x)/n! + o(h^n)$ as $h\quad 0$.

If (1) holds as $h\quad 0^+$ then we say that f has an n th Peano derivative from the right at x and denote the numbers instead by $f_+\ (x), \cdots, f_{n+}\ (x)$.

If f has an $(n - 1)$ th Peano derivative at x and if

$$(2)\qquad \lim_{h} \frac{f(x + h) - f(x) - \cdots - h^{n-} f_{n-}(x)/(n-1)!}{h^n/n!} = + ,$$

then we write $f_n(x) = +$. We define $f_n(x) = -$ in a similar way. Furthermore $f_{n+}(x) = +$ or $-$ is defined by letting $h\quad 0$ in (2).

Theorem 1: If f has an n th Peano derivative, $f_n(x)$, at each x in $\mathbb{R}$ with infinite values allowed, then f_n is a function of Baire class one.

(This theorem originally appeared in [1] but with an invalid proof.)

To establish further properties of such functions f_n the following auxiliary theorem is useful and of interest in its own right.

Theorem 2: If $f_n(x)$ exists for all x in $\mathbb{R}$ with infinite values allowed, and if f_n is bounded above or below on an interval I, then $f_n = f^{(n)}$, the ordinary n th derivative of f, on I.

This result can be established by copying the proof of the corresponding assertion for the finite case from [2], [4] or [5] and making the necessary minor changes. We chose the last of these three since it required only a small modification in a lemma.

Using Theorem 2 we establish the following properties of Peano derivatives.

Theorem 3: Let n 2 and suppose $f_n(x)$ exists for all x in $\mathbb{R}$ with infinite values allowed. Then

(i) f_n has the Darboux property

and

(ii) f_n has the Denjoy-Clarkson property.

The proof of Theorem 3 uses Theorem 2 with a theorem from [3] for (i) and one from [6] for (ii). These two theorems are stated only for finite functions but in both cases it is easily seen that they hold for extended real-valued functions as well. The assumption n 2 is needed only for (i) which is false for $n = 1$. The statement (i) is true for $n = 1$.

The only positive result for unilateral Peano derivatives is the following one for the finite case.

Theorem 4: If $f_{n+}(x)$ exists and is finite for each x in $\mathbb{R}$, then f_{n+} is of Baire class one.

The proof of this theorem uses the following lemma which has independent interest.

Lemma: If f is as in Theorem 4, then $f, f_{+}, \cdots, f_{n-)+}$ are Baire* one functions; that is, each nonempty, perfect set contains a portion relative to which they are continuous.

We conclude with an example showing that the assumption of finite in Theorem 4 is essential.

Example: There is a function g defined on $[0,1]$ such that:

(a) g is bounded and approximately continuous

(b) $g'_{+}(x)$ exists for every x in $[0,1]$ allowing infinite values

and

(c) g'_{+} is not of Baire class one on $[0,1]$.

By (a) there is a continuous function f such that $f' = g$ on $[0,1]$. Continuing, for any n 2 there is a continuous function f such that $f^{(n-)} = g$ and hence $f_+^{(n)} = g_+'$.

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