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ON CATEGORY PROJECTIONS OF CARTESIAN PRODUCT A×A.

We use the terminology introduced in [1]. If $f:R \to R$ and $E \subseteq R^2$ we say that f-projection of E is the set $\{c: (f+c) \cap E \neq \emptyset\}$, that is the set of all c for which the graph of f+c intersects E. The f-category projection of E is the set $\{c: \mathrm{dom}[(f+c) \cap E]$ is of second category$\}$. We use the word projection where f is linear.

In [1] J.Ceder and D.K.Ganguly proved that there exists a second category set A such that the projection of A×A onto any line with rational slope and rational intercept does not contain an interval. Next is submitted the following question: " It is unknown whether or not a second category set A can be found such that the /category/ projection of A×A fails to have a non-empty interior in each direction." .
Main result with respect to this problem is following.

THEOREM. If Martin's Axiom is assumed that there exists a second category set A such that the category projection of A×A onto each line has empty interior.

P r o o f. Let Q denote the set of rational numbers. Let $\{G_\alpha\}_{\alpha<c}$ be a well-ordering of all residual G_δ- subsets of the line and $\{r_\xi\}_{\xi<c}$ be a well-ordering of all real numbers such that $r_0 = 0$. Choose $a_0 = \min\{r_\xi : r_\xi \in G_0 - Q\}$. Suppose we have chosen $a_\alpha \in G_\alpha$

for all $\alpha<\gamma$. Notice that the set

$$H_\gamma = G_\gamma - \bigcup_{\alpha,\beta<\gamma}\left(\{r_\alpha a_\beta + Q\} \cup \{r_\alpha^{-1}(a_\beta + Q) : \alpha>0\}\right)$$ is non-empty.

Choose $a_\gamma = \min\{r_\xi : r_\xi \in H_\gamma\}$. Put $A = \{a_\gamma : \gamma<c\}$. The set A is of second category because it intersects each residual G_δ set.

Let r_α be a fix direction. We shall prove that for any $q \in Q - \{0\}$ the cardinality of $B = \{(x,y) : y = r_\alpha x + q\} \cap A \times A$ is less than continuum and /since MA holds/ B is of the first category.

Assume that $a_\gamma = r_\alpha a_\beta + q$. Then the following three cases may happen.

1/ $\gamma>\beta$. Then $\gamma\leq\alpha$ and the cardinality of the set $\{(a_\beta, a_\gamma) : a_\gamma = r_\alpha a_\beta + q, \gamma>\beta\}$ is less than continuum.

2/ $\gamma<\beta$. Then the identity $a_\gamma = r_\alpha a_\beta + q$ is equivalent of the identity $a_\beta = r_\alpha^{-1}(a_\gamma - q)$ /since $r_\alpha \neq 0$ /. Thus $\beta\leq\alpha$ and the cardinality of the set $\{(a_\beta, a_\gamma) : a_\gamma = r_\alpha a_\beta + q, \gamma<\beta\}$ is less than continuum.

3/ $\gamma=\beta$. Then the set $\{(a_\beta, a_\beta) : a_\beta = r_\alpha a_\beta + q\}$ has at most one element.

Thus for any $\alpha< c$ and $q \in Q - \{0\}$ the intersection line of the form $y = r_\alpha x + q$ and $A\times A$ is of the first category.

Let f be a line of form $y = rx + v$, for $r, v \in R$, B be the f-category projection of $A\times A$ and $C = -v + Q$. Then C is dense in R and $B \cap C = \emptyset$. Thus B has empty interior.

References. [1] J.Ceder and D.K.Ganguly, On projections of big planar sets, Real Analysis Exchange, Vol.9, No. 1, /1983-84/, 206-214.

The following four questions are submitted in connection with the article by S. J. Agronsky in the Proceedings of the Eighth Summer Symposium Section of this issue of the Exchange.

174. It is known that for each $f \in C([0,1])$, there exists $g \in C^1([0,1])$ such that $\{x:f(x) = g(x)\}$ is uncountable. Can this result be improved to g twice differentiable or $g \in C^n([0,1])$?

175. It is known that there exists $f \in C([0,1])$ such that $\{x:f(x) = g(x)\}$ is finite for all g analytic. Must such f be well-behaved, e.g. in $C^\infty([0,1])$?

176. It is known that for all $f \in C(0,1])$, there exists $g \in C^\infty([0,1])$ such that $\{x:f(x) = g(x)\}$ is infinite. Can the continuity requirement of f be weakened (e.g. to f Darboux or even arbitrary)?

177. Let P_n denote the polynomials (in one real variable) of degree $\leq n$. For each pair (n,k) where $n = 2,3,\cdots$ and $k = n + 2, n + 3, \cdots$ suppose $f \in C([0,1])$ and $\text{card}\{x:f(x) = p(x)\} \leq k$ for all polynomials $p \in P_n$. Find the number $s(n,k)$ such that $[0,1]$ can be decomposed into $s(n,k)$ subintervals on each of which f is $n + 1$ concave of convex.