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SECOND CATEGORY E WITH EACH PROJ($\mathbb{R}^2 \setminus E^2$) DENSE

THEOREM. The continuum hypothesis (CH) implies that there exists a set $E \subseteq \mathbb{R}$ such that (i) E has non-empty intersection with every uncountable closed set and (ii) the projection of $E\times E$ on any line contains no interval.

REMARK. Property (i) implies that E is both of full outer measure and of second category in every nondegenerate interval. Property (ii) is equivalent to the existence of a dense set of lines in every direction, disjoint from $E\times E$.

PROOF OF THE THEOREM. By CH, we can list as F_α, $0 \leq \alpha < \omega_1$, all uncountable closed subsets of $\mathbb{R}$, and as (ℓ_α, I_α), $0 \leq \alpha < \omega_1$, all pairs consisting of a line through the origin, not one of the axes, and a non-degenerate interval on it. By an α-line we shall mean a perpendicular to ℓ_α through a point of I_α.

By transfinite induction we shall define for $0 \leq \alpha < \omega_1$ an α-line k_α and a point $x_\alpha \in F_\alpha$ such that

$$K_\alpha \cap (E_\alpha \times E_\alpha) = \phi, \qquad \ldots (1)$$

where $K_\alpha = \cup\{k_\beta : \beta \leq \alpha\}$ and $E_\alpha = \{x_\beta : \beta \leq \alpha\}$. Let x_0 be any point of F_0 and k_0 any 0-line not through (x_0, x_0); condition (1) is then satisfied for $\alpha=0$.

Now suppose that k_α, x_α have been defined for all $\alpha<\gamma$, so as to satisfy the above conditions. Write $E'_\gamma = \{x_\beta : \beta<\gamma\}$, and choose as k_γ any γ-line for which $k_\gamma \cap (E'_\gamma \times E'_\gamma) = \phi$: this is possible because it requires the avoidance of only countably many lines (those through points of $E'_\gamma \times E'_\gamma$, perpendicular to ℓ'_γ). We shall then have

$$K_\gamma \cap (E'_\gamma)^2 = [\cup\{K_\alpha \cap (E_\alpha)^2 : \alpha<\gamma\}] \cup [k_\gamma \cap ([E'_\gamma)^2] = \phi. \qquad \ldots (2)$$

Choose as x_γ any point of F_γ such that (1), with α replaced by γ, becomes true: this is possible because in view of (2) we have

$$K_\gamma \cap (E_\gamma)^2 = [K_\gamma \cap \{(x_\gamma, x_\gamma)\}] \cup [K_\gamma \cap (\{x_\gamma\} \times E'_\gamma)] \cup [K_\gamma \cap (E'_\gamma \times \{x_\gamma\})],$$

and to make each of the three sets on the right hand side empty requires the avoidance of only countably many points (those x's such that (x,0) or (0,x) is in the projection on an axis of a point of intersection of a line of K_γ with the line x = y or a line of $\mathbb{R} \times E'_\gamma$ or $E'_\gamma \times \mathbb{R}$).

Let $E = \{x_\alpha : 0 \leq \alpha < \omega_1\}$. Property (i) holds because $x_\alpha \in E \cap F_\alpha$. Since

$$(\cup\{k_\beta : \beta<\omega_1\}) \cap (E \times E) = \cup_\alpha [K_\alpha \cap (E_\alpha \times E_\alpha)] = \phi$$

by (1), therefore E×E has the projection property (ii) (for all lines other than the axes, and hence also for the axes).